University of Anbar
College of Engineering
Mechanical Engineering Dept.

Advanced Heat Transfer/ I
Conduction and Radiation

Handout Lectures for MSc. / Power
Chapter One/ Introductory Concepts

Course Tutor

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Chapter One
Introductory Concepts

1.1 Modes of Heat Transfer

Heat transfer (or **heat**) is thermal energy in transit due to a spatial temperature difference.

Whenever a temperature difference exists in a medium or between media, heat transfer must occur.

As shown in Figure 1.1, we refer to different types of heat transfer processes as modes. When a temperature gradient exists in a stationary medium, which may be a solid or a fluid, we use the term "***conduction***" to refer to the heat transfer that will occur across the medium. In contrast, the term "***convection***" refers to heat transfer that will occur between a surface and a moving fluid when they are at different temperatures. The third mode of heat transfer is termed "***thermal radiation***". All surfaces of finite temperature emit energy in the form of electromagnetic waves. Hence, in the absence of an intervening medium, there is net heat transfer by radiation between two surfaces at different temperatures.

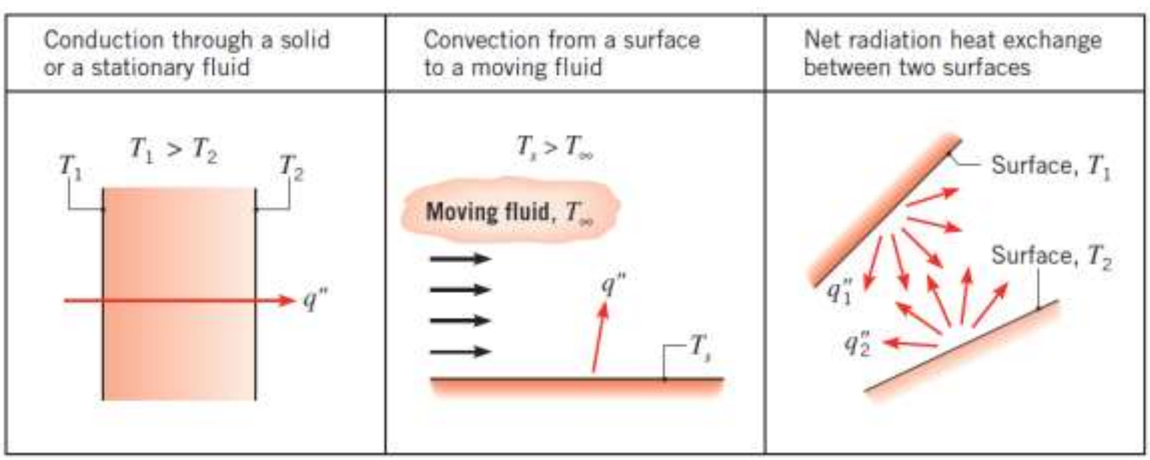


Figure 1.1: Conduction, convection, and radiation heat transfer modes.

As engineers, it is important that we understand the physical mechanisms which underlie the heat transfer modes and that we be able to use the rate equations that quantify the amount of energy being transferred per unit time.

1.1.1 Conduction Heat Transfer

At mention of the word *conduction*, we should immediately conjure up concepts of *atomic* and *molecular activity* because processes at these levels sustain this mode of heat transfer. Conduction may be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interactions between the particles.

The physical mechanism of conduction is most easily explained by considering a gas and using ideas familiar from your thermodynamics background. Consider a gas in which a temperature gradient exists, and assume that there is *no bulk, or macroscopic, motion.* The gas may occupy the space between two surfaces that are maintained at different temperatures, as shown in Figure 1.2. We associate the temperature at any point with the energy of gas molecules in proximity to the point. This energy is related to the random translational motion, as well as to the internal rotational and vibrational motions, of the molecules.

Higher temperatures are associated with higher molecular energies. When neighboring molecules collide, as they are constantly doing, a transfer of energy from the more energetic to the less energetic molecules must occur. In the presence of a temperature gradient, energy transfer by conduction must then occur in the direction of decreasing temperature. This would be true even in the absence of collisions, as is evident from Figure 1.2. The hypothetical plane at is constantly being crossed by molecules from above and below due to their *random* motion. However, molecules from above are associated with a higher temperature than those from below, in which case there must be a *net* transfer of energy in the positive *x*-direction. Collisions between molecules enhance this energy transfer.

We may speak of the net transfer of energy by random molecular motion as a *diffusion* of energy.

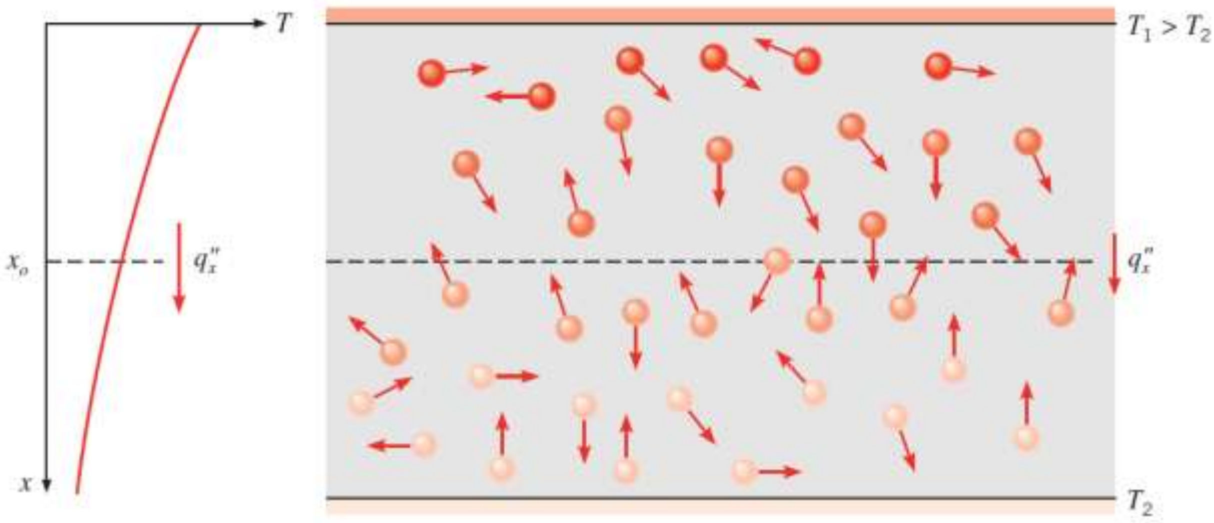


Figure 1.2: Association of conduction heat transfer with diffusion of energy due to molecular activity.

The situation is much the same in liquids, although the molecules are more closely spaced and the molecular interactions are stronger and more frequent. Similarly, in a solid, conduction may be attributed to atomic activity in the form of lattice vibrations. The modern view is to ascribe the energy transfer to *lattice waves* induced by atomic motion. In an electrical nonconductor, the energy transfer is exclusively via these lattice waves; in a conductor, it is also due to the translational motion of the free electrons.

Examples of conduction heat transfer are legion. The exposed end of a metal spoon suddenly immersed in a cup of hot coffee is eventually warmed due to the conduction of energy through the spoon. On a winter day, there is significant energy loss from a heated room to the outside air. This loss is principally due to conduction heat transfer through the wall that separates the room air from the outside air.

Heat transfer processes can be quantified in terms of appropriate *rate equations*. These equations may be used to compute the amount of energy being transferred per unit time. For heat conduction, the rate equation is known as *Fourier's law*. For the one-dimensional plane wall shown in Figure 1.3, having a temperature distribution $T(x)$, the rate equation is expressed as,

$$q_x^{"} = -k\frac{dT}{dx} \quad (1\text{-}1)$$

The heat flux (W/m^2) is the ***heat transfer rate*** in the x-direction per unit area perpendicular to the direction of transfer, and it is proportional to the ***temperature gradient***, dT/dx, in this direction. The parameter k is a transport property known as the ***thermal conductivity*** (W/mK) and is a characteristic of the wall material. The ***minus sign*** is a consequence of the fact that heat is transferred in the direction of decreasing temperature. Under the steady-state conditions shown in Figure 1.3, where the temperature distribution is linear, and the temperature gradient may be expressed as,

$$q_x^{"} = -k\frac{dT}{dx} = -k\,\frac{T_2 - T_1}{L} = k\,\frac{T_1 - T_2}{L} \quad (1\text{-}2)$$

The *heat rate* by conduction, q_x (W), through a plane wall of area A is then the product of the flux and the area, $q_x = q_x^{"} \times A$

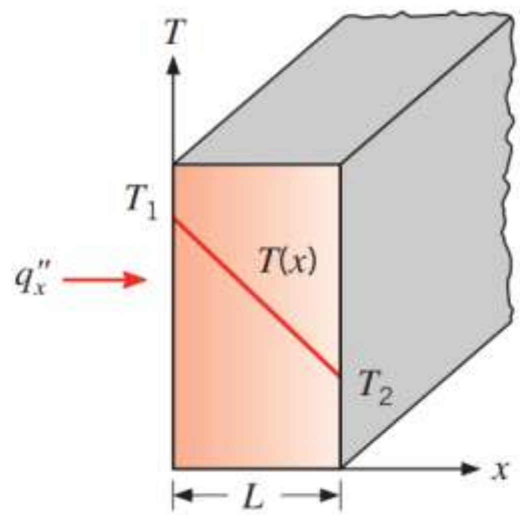


Fig. 1-3: One-dimensional heat transfer by conduction

1.1.2 Convection Heat Transfer

The convection heat transfer mode is comprised of *two mechanisms*. In addition to *energy transfer* due to *random molecular motion* (diffusion), energy is also transferred by the *bulk*, or *macroscopic*, *motion of the fluid*. This fluid motion is associated with the fact that, at any instant, large numbers of molecules are moving collectively or as aggregates. Such motion, in the presence of a temperature gradient, contributes to heat transfer. Because the molecules in the aggregate retain their random motion, the total heat transfer is then due to a superposition of energy transport by the random motion of the molecules and by the bulk motion of the fluid. The term *convection* is customarily used when referring to this cumulative transport and the term *advection* refers to transport due to bulk fluid motion

Convection heat transfer may be classified according to the nature of the flow. We speak of *forced convection* when the flow is caused by *external means*, such as by a fan, a pump, or atmospheric winds. As an example, consider the use of a fan to provide forced convection air cooling of hot electrical components on a stack of printed circuit boards (Figure 1.4a). In contrast, for *free* (or *natural*) *convection*, the flow is induced by *buoyancy forces*, which are due to density differences caused by temperature variations in the fluid. An example is the free convection heat transfer that occurs from hot components on a vertical array of circuit boards in air (Figure 1.4b). Air that makes contact with the components experiences an increase in temperature and hence a reduction in density. Since it is now lighter than the surrounding air, buoyancy forces induce a vertical motion for which warm air ascending from the boards is replaced by an inflow of cooler ambient air.

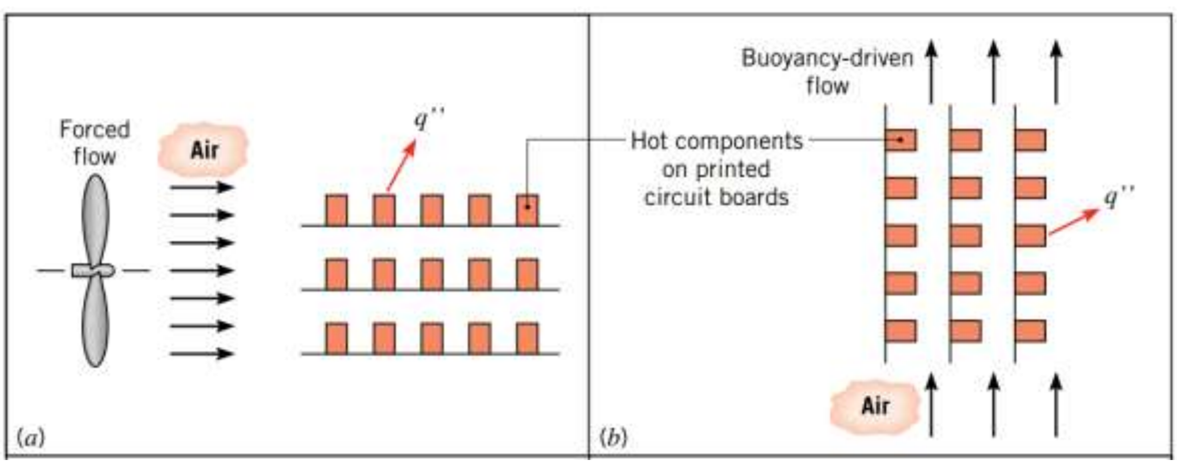


Fig. 1-4: Convection heat transfer processes. (a) Forced convection. (**b**) Natural convection.

Regardless of the nature of the convection heat transfer process, the appropriate rate equation is of the form,

$$q'' = h\,(T_s - T_\infty) \qquad (1\text{-}3)$$

where , the convective heat flux (W/m^2), is proportional to the difference between the surface and fluid temperatures, T_s and T_∞, respectively. This expression is known as *Newton's law of cooling*, and the parameter h (W/m^{2}K) is termed the *convection heat transfer coefficient*. This coefficient depends on conditions in the boundary layer, which are influenced by surface geometry, the nature of the fluid motion, and an assortment of fluid thermodynamic and transport properties. In the solution of such problems we presume h to be known, using typical values given in Table 1.1.

Table 1.1: Typical values of the convection heat transfer coefficient.

Process	*h* (W/m^2 · K)
Free convection	
Gases	2–25
Liquids	50–1000
Forced convection	
Gases	25–250
Liquids	100–20,000
Convection with phase change	
Boiling or condensation	2500–100,000

1.1.3 Radiation Heat Transfer

Thermal radiation is *energy emitted* by matter that is at a nonzero temperature. Although we will focus on radiation from solid surfaces, emission may also occur from liquids and gases. Regardless of the form of matter, the emission may be attributed to changes in the electron configurations of the constituent atoms or molecules. The energy of the radiation field is transported by electromagnetic waves (or alternatively, photons). While the transfer of energy by conduction or convection requires the presence of a material medium, radiation does not. In fact, radiation transfer occurs most efficiently in a vacuum. Consider radiation transfer processes for the surface of Figure 1.5a. Radiation that is emitted by the surface originates from the thermal energy of matter bounded by the surface, and the rate at which energy is released per unit area (W/m^2) is termed the surface *emissive power*, *E*. There is an upper limit to the *emissive power*, which is prescribed by the *Stefan Boltzmann* law:

$$E_b = \sigma T_s^4 \qquad (1\text{-}4)$$

where T_s is the absolute temperature (K) of the surface and σ is the *Stefan Boltzmann constant* ($\sigma = 5.67 \times 10^{-8}\ W/m^2K^4$). Such a surface is called an *ideal radiator* or *blackbody*. The heat flux emitted by a *real surface* is less than that of a blackbody at the same temperature and is given by,

$$E = \varepsilon \sigma T_s^4 \qquad (1\text{-}5)$$

Where ε is a radiative property of the surface termed the emissivity. With values in the range $0 \le \varepsilon \le 1$, this property provides a measure of how efficiently a surface emits energy relative to a blackbody.

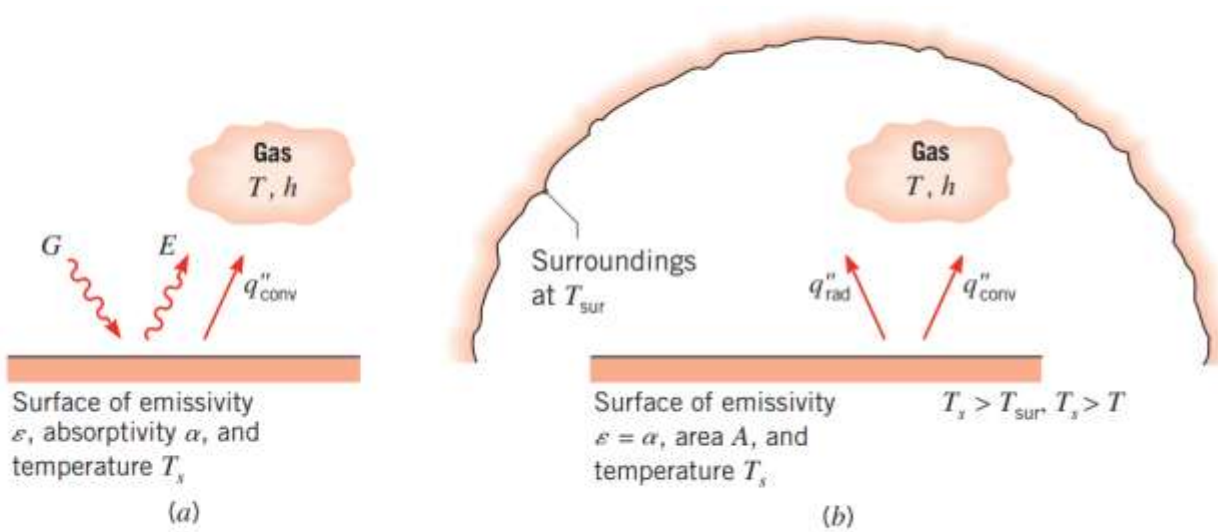


Fig. 1-5: Radiation exchange: (**a**) at a surface and (**b**) between a surface and large surroundings.

A special case that occurs frequently involves radiation exchange between a small surface at T_s and a much larger, isothermal surface that completely surrounds the smaller one (Figure 1.5b). The surroundings could, for example, be the walls of a room or a furnace whose temperature T_{sur} differs from that of an enclosed surface ($T_{sur} \neq T_s$). For such a condition, the irradiation may be approximated by emission from a blackbody at T_{sur}, in which case $G = \sigma T_{sur}^4$. If the surface is assumed to be one for which $\alpha = \varepsilon$ (a *gray* surface), the net rate of radiation heat transfer from the surface, expressed per unit area of the surface, is

$$q''_{rad} = \frac{q}{A} = \varepsilon E_b(T_s) - \alpha G = \varepsilon\sigma(T_s^4 - T_{sur}^4) \qquad (1\text{-}6)$$

This expression provides the difference between thermal energy that is released due to radiation emission and that gained due to radiation absorption.

A surface which absorbs all radiation incident upon it (α=1) or at a specified temperature emits the maximum possible radiation is called (black surface). The emissivity of a surface, ε, is defined as;

$$\frac{q}{q_b} = \varepsilon$$

where (q and q_b) are the radiant heat fluxes from this surface and from a black surface respectively at the same temperature. Under thermal equilibrium (α=ε) for all surfaces (*Kirchhoff's law*).

When two bodies exchange heat by radiation, the net heat exchange is given by Stefan-Boltzmann's law of radiation which was found experimentally by Stefan and later proved thermodynamically by Boltzmann. Thus;

$$q = F_G \varepsilon\sigma(T_s^4 - T_{sur}^4)$$

Where F_G is geometric view factor, configuration factor or shape factor.

1.2 Fourier's law of conduction

Microscopic theories such as the *kinetic theory of gases* and the *free-electron theory of metals* have been developed to the point where they can be used to predict conduction through media. However, the *macroscopic* or *continuum* theory of conduction, which is the subject matter of this course, disregards the molecular structure of continua. Thus conduction is taken to be phenomenological and its effects are determined by experiment as described in details in *Section **1.1.1***.

The molecular structure of material (continua) may be classified according to variations in thermal conductivity. A material (continuum) is said to be ***homogeneous*** if its conductivity does not vary from point to point within the continuum, and ***heterogeneous*** if there is such variation. Furthermore, continua in which the conductivity is the same in all directions are said to be ***isotropic***, whereas those in which there exists directional variation of conductivity are said to be ***anisotropic***. Some materials consisting of a fibrous structure exhibit anisotropic character, for example, wood and asbestos. Materials having a porous structure, such as wool or cork, are examples of heterogeneous continua. In this course, except where explicitly stated otherwise, we shall be studying only the problems of isotropic continua. Because of the symmetry in the conduction of heat in isotropic continua, the flux of heat at a point must be normal to the isothermal surface through this point.

According to the *first law of thermodynamics*, under steady conditions there must be a constant rate of heat q through any cross section of the geometry (such, walls, cylinders and spheres). From the *second law of thermodynamics* we know that the direction of this heat is from the higher temperature to the lower. Therefore, equations 1.1 and 1.2 give Fourier's law for ***homogeneous isotropic continua***. Equations (1.1 and 1.2) may also be used for a fluid (liquid or gas) placed between two plates a distance L apart, provided that suitable precautions are taken to eliminate convection and radiation. Therefore, equations (1.1 and 1.2) describe the conduction of heat in fluids as well as in solids.

Let the temperatures of two isothermal surfaces corresponding to the locations *x* and *x*+Δ*x* be *T* and *T*+Δ*T*, respectively (Figure 1.6). Since this plate may be assumed to be locally homogeneous, equation (1.1) can be used for a layer of the plate having the thickness Δ*x* as Δ*x*+0. Thus it becomes possible to state the differential form of Fourier's law of conduction, giving the heat flux at *x* in the direction of increasing *x*, as follows:

$$q_x = -k \lim_{\Delta x \to 0} \left(\frac{\Delta T}{\Delta x}\right) = -k \frac{\partial T}{\partial x} \qquad (1\text{-}7)$$

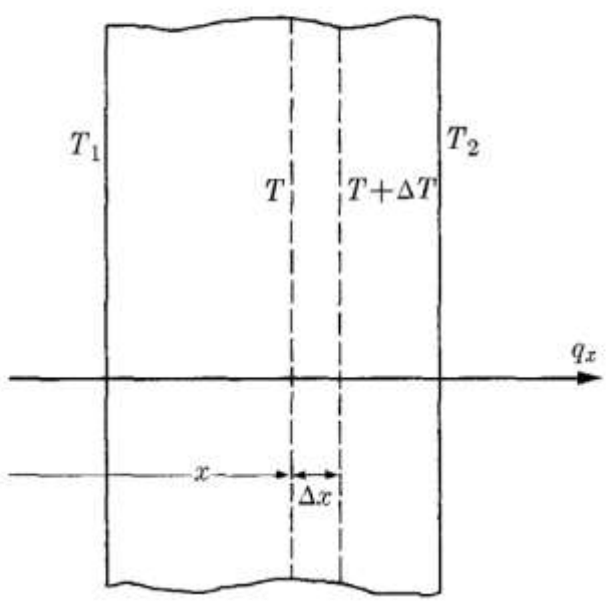


Fig. 1-6: Isothermal surfaces.

Fourier's law for heterogeneous isotropic continua. In equation (1.7), by introducing a minus sign we have made q_x, positive in the direction of increasing *x*. It is important to note that this equation is independent of the temperature distribution. Thus, for example, in figure 1.7 (a) $\frac{\partial T}{\partial x} < 0$ and q_x, > 0, whereas in figure 1.7 (b) $\frac{\partial T}{\partial x} > 0$ and q_x, < 0. Both results agree with the second law of thermodynamics in that the heat diffuses from higher to lower temperatures.

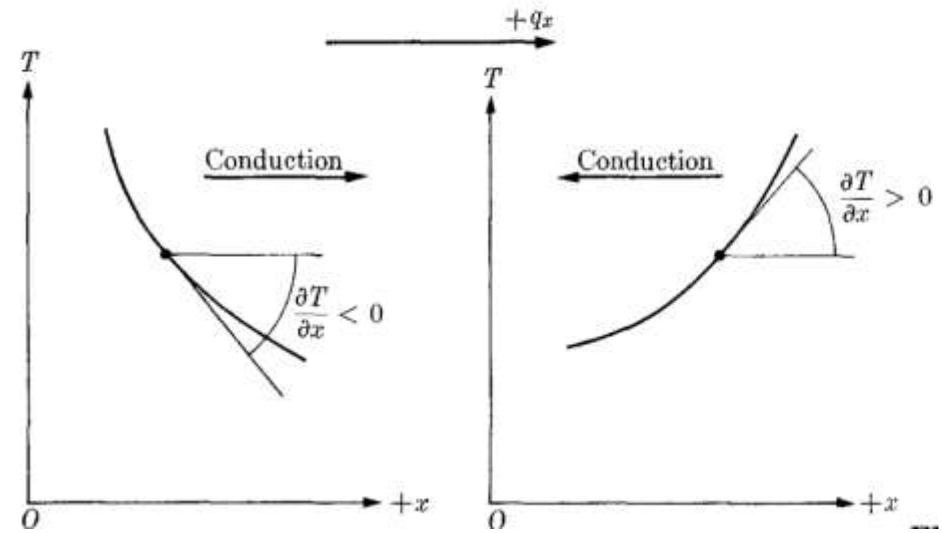


Fig. 1-7: Independent of the temperature distribution.

Equation (1.7) may be readily extended to any isothermal surface if we state that the heat flux across an isothermal surface is

$$q_n = -k\frac{\partial T}{\partial n} \qquad (1\text{-}8)$$

Generalizing Fourier's law for isotropic continua, we may assume each component of the heat flux vector to be linearly dependent on all components of the temperature gradient at the point. Thus, for example, the Cartesian form of *Fourier's law for heterogeneous anisotropic continua* becomes

$$\left.\begin{aligned} q_x &= -(k_{11}\frac{\partial T}{\partial x} + k_{12}\frac{\partial T}{\partial y} + k_{13}\frac{\partial T}{\partial z}) \\ q_y &= -(k_{21}\frac{\partial T}{\partial x} + k_{22}\frac{\partial T}{\partial y} + k_{23}\frac{\partial T}{\partial z}) \\ q_z &= -(k_{31}\frac{\partial T}{\partial x} + k_{32}\frac{\partial T}{\partial y} + k_{33}\frac{\partial T}{\partial z}) \end{aligned}\right\} \qquad (1\text{-}9)$$

The value of k for a continuum depends in general on the chemical composition, the physical state, and the structure, temperature, and pressure. In solids the pressure dependency, being very small, is always neglected. For narrow temperature intervals the temperature dependency may also be negligible. Otherwise a linear relation is assumed in the form

$$k = k_o(1 + \beta T) \tag{1-10}$$

where β is small and negative for most solids.

1.3 Equation of Conduction

A major objective in a conduction analysis is to determine the temperature field in a medium resulting from conditions imposed on its boundaries. That is, we wish to know the temperature distribution, which represents how temperature varies with position in the medium. Once this distribution is known, the conduction heat flux at any point in the medium or on its surface may be computed from Fourier's law. Other important quantities of interest may also be determined.

Consider a homogeneous medium within which there is no bulk motion (*advection*) and the temperature distribution $T(x, y, z)$ is expressed in **Cartesian coordinates**. By applying conservation of energy, we first define an infinitesimally small (*differential*) control volume, *dx.dy.dz*, as shown in figure 1.8. Choosing to formulate the first law at an instant of time, the second step is to consider the energy processes that are relevant to this control volume. In the absence of motion (or with uniform motion), there are no changes in mechanical energy and no work being done on the system. Only thermal forms of energy need be considered. Specifically, if there are temperature gradients, conduction heat transfer will occur across each of the control surfaces. The conduction heat rates perpendicular to each of the control surfaces at the *x*-, *y*-, and *z*-coordinate locations are indicated by the terms q_x, q_y and q_z, respectively.

- ✓ The conduction heat rates at the opposite surfaces can then be expressed as a Taylor series expansion where, neglecting higher-order terms,

$$\left.\begin{aligned} q_{x+dx} &= q_x + \frac{\partial q_x}{\partial x}\,dx \\ q_{y+dy} &= q_y + \frac{\partial q_y}{\partial y}\,dy \\ q_{z+dz} &= q_z + \frac{\partial q_z}{\partial z}\,dz \end{aligned}\right\} \tag{1-11}$$

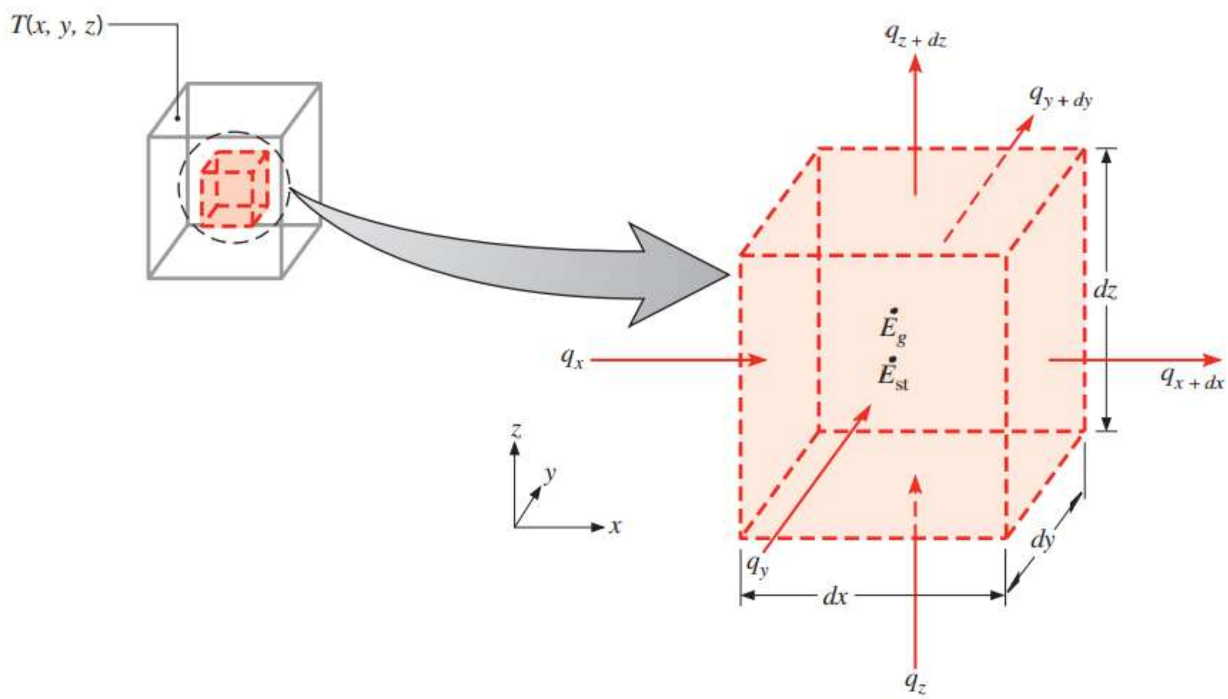


Fig. 1-8: Differential control volume, *dx dy dz*, for conduction analysis in Cartesian coordinates.

- ✓ Within the medium there may also be an energy source term associated with the rate of thermal energy generation. This term is represented as:

$$E_g = \dot{g}\,dxdydz \tag{1-12}$$

Where $\dot{g}$ is the rate at which energy is generated per unit volume of the medium (W/m^3).

- ✓ Changes may occur in the amount of the internal thermal energy stored by the material in the control volume and the energy storage term may be expressed as:

$$E_{st} = \rho C_p \frac{\partial T}{\partial t}\, dxdydz \qquad (1\text{-}13)$$

where $\rho C_p \frac{\partial T}{\partial t}$ is the time rate of change of the sensible (thermal) energy of the medium per unit volume.

On a rate basis, the general form of the conservation of energy requirement is:

$$E_{in} + E_g - E_{out} = E_{st} \qquad (1\text{-}14)$$

Hence, recognizing that the conduction rates constitute the energy inflow E_{in} and outflow E_{out}, and substituting Equations 1.12 and 1.13into Equation 1.14, we obtain,

$$q_x + q_y + q_z + \dot{g}\, dxdydz - q_{x+dx} - q_{y+dy} - q_{z+dz} = \rho C_p \frac{\partial T}{\partial t}\, dxdydz \qquad (1\text{-}15)$$

The conduction heat rates in an isotropic material may be evaluated from Fourier's law,

$$\left.\begin{aligned} q_x &= -k dydz \frac{\partial T}{\partial x} \\ q_y &= -k dxdz \frac{\partial T}{\partial y} \\ q_z &= -k dxdy \frac{\partial T}{\partial z} \end{aligned}\right\} \qquad (1\text{-}16)$$

Substituting Equation 1.16 into Equation 1.15 and dividing out the dimensions of the control volume (*dx dy dz*), we obtain

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{g} = \rho C_p \frac{\partial T}{\partial t} \qquad (1\text{-}17)$$

Equation 1.17 is the general form, in Cartesian coordinates, of the *heat diffusion* equation. This equation, often referred to as the heat equation, provides the basic tool for heat conduction analysis. Equation 1.17, therefore states that at any point in the medium the *net rate of energy transfer by conduction* into a unit volume plus the *volumetric rate of thermal energy generation* must equal the *rate of change of thermal energy stored* within the volume.

If the **thermal conductivity is constant**, the heat equation is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (1\text{-}18)$$

where α $(=\frac{Heat\ conducted}{Heat\ stored}=\,{}^{k}/_{\rho\,C_p})$ is the thermal diffusivity.

- ✓ The heat equation under constant the thermal conductivity and steady-state conditions is called ***Poisson Equation*** as,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{g}}{k}=0 \quad (1\text{-}18\text{-}a)$$

- ✓ The heat equation under constant the thermal conductivity and no heat generation is called ***Diffusion Equation*** as,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (1\text{-}18\text{-}b)$$

- ✓ The heat equation under constant the thermal conductivity, no heat generation and steady-state conditions is called ***Laplace Equation*** as,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \quad (1\text{-}18\text{-}c)$$

Under **steady-state conditions**, there can be no change in the amount of energy storage; hence Equation 1.17 reduces to

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{g}=0 \quad (1\text{-}19)$$

Moreover, if the heat transfer is *one-dimensional* (e.g., in the *x*-direction) and there is *no energy generation*, Equation 1.19 reduces to

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \quad (1\text{-}20)$$

Cylindrical Coordinates

The heat equation may also be expressed in cylindrical coordinates. The differential control volume for this coordinate system is shown in Figure 1.9. In cylindrical coordinates, Fourier's law is

$$\left.\begin{aligned} q_r'' &= -k\frac{\partial T}{\partial r} \\ q_\phi'' &= -\frac{k}{r}\frac{\partial T}{\partial \phi} \\ q_z'' &= -k\frac{\partial T}{\partial z} \end{aligned}\right\} \qquad (1\text{-}21)$$

Where, $x = r\ \mathrm{Cos}\phi$, $y = r\ \mathrm{Sin}\phi$, $z = z$

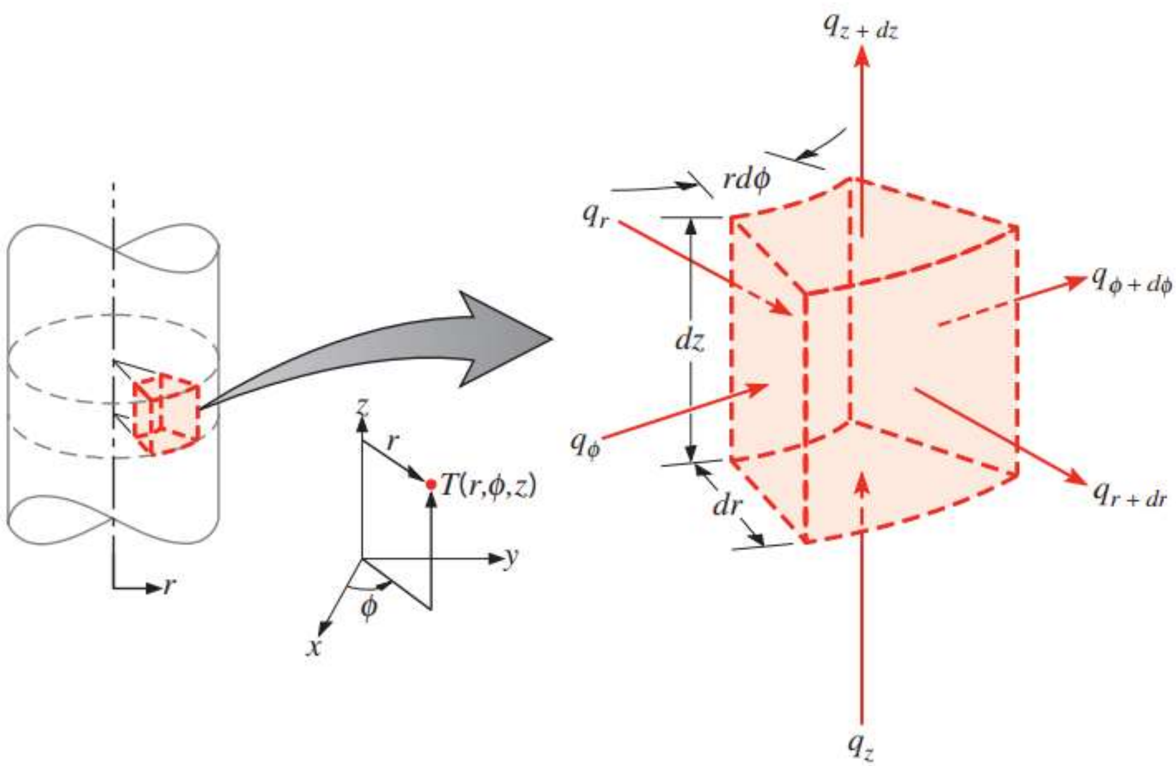


Fig. 1-9: Differential control volume, dr rd ϕ dz, for conduction analysis in cylindrical coordinates (r, ϕ, z).

Applying an energy balance to the differential control volume of Figure 1.9, the following general form of the heat equation is obtained:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C_p\frac{\partial T}{\partial t} \qquad (1\text{-}22)$$

If the **thermal conductivity is constant**, the heat equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 T}{\partial \phi^2}\right) + \left(\frac{\partial^2 T}{\partial z^2}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (1\text{-}23)$$

Spherical Coordinates

The heat equation may also be expressed in spherical coordinates. The differential control volume for this coordinate system is shown in Figure 1.10. In spherical coordinates, Fourier's law is

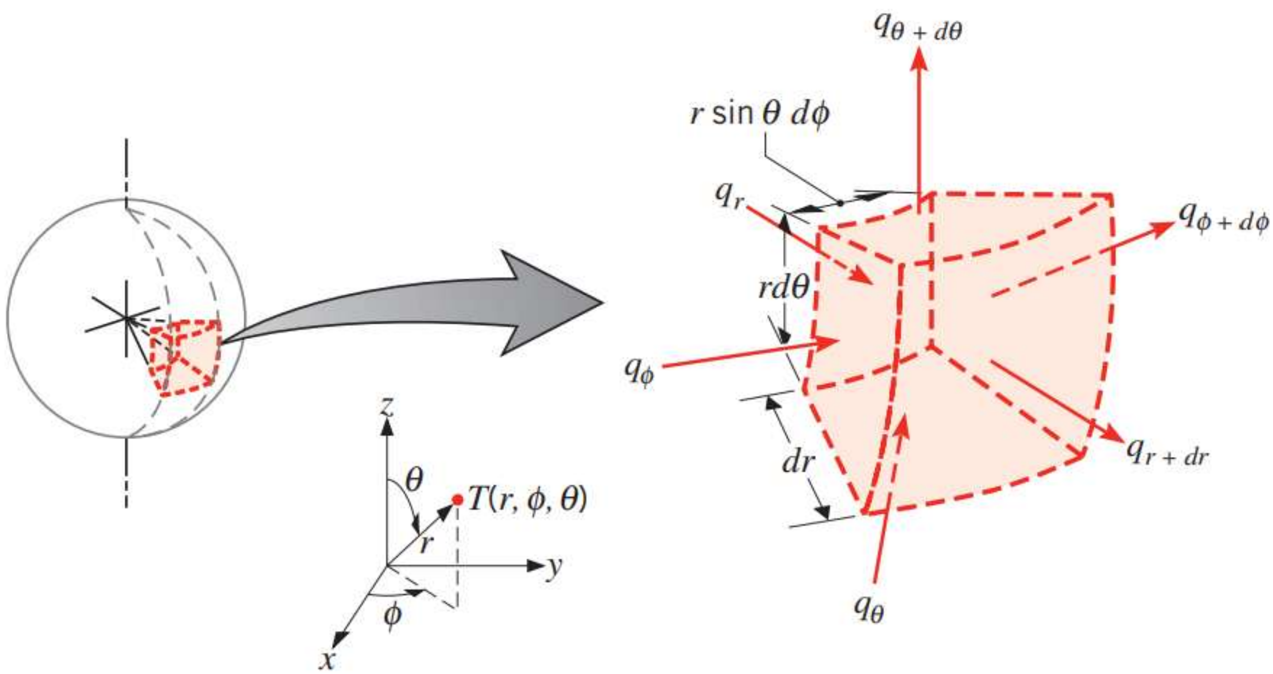


Fig. 1-10: Differential control volume, *dr. r sinθ dϕ. rdθ*, for conduction analysis in spherical coordinates (*r, θ, ϕ*).

$$\left.\begin{aligned} q_r'' &= -k\frac{\partial T}{\partial r} \\ q_\phi'' &= -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi} \\ q_\theta'' &= -\frac{k}{r}\frac{\partial T}{\partial \theta} \end{aligned}\right\} \quad (1\text{-}24)$$

Where, $x = r\ \text{Cos}\phi\ \text{Sin}\theta$, $y = r\ \text{Sin}\phi\ \text{Sin}\theta$, $z = r\ \text{Cos}\theta$

Applying an energy balance to the differential control volume of Figure 1.10, the following general form of the heat equation is obtained:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{g}=\rho C_p\frac{\partial T}{\partial t} \quad (1\text{-}25)$$

If the **thermal conductivity is constant**, the heat equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (1\text{-}26)$$

Exercise 1:

Derive the general 3D heat conduction equation through isotropic media in cylindrical and spherical coordinates using: Coordinate transformation and Energy balance for a finite volume element.

1.4 Boundary and Initial Conditions

1.4.1 Boundary (surface) conditions:

The most frequently encountered boundary conditions in conduction are as follows,

A. Prescribed temperature

The surface temperature of the boundaries is specified to be a constant or a function of space and/or time.

B. Prescribed heat flux

The heat flux across the boundaries is specified to be a constant or a function of space and/or time. The mathematical description of this condition may be given in the light of Kirchhoff's current law; that is, the algebraic sum of heat fluxes at a boundary must be equal to zero. Hereafter the sign is to be assumed positive for the heat flux to the boundary and negative for that from the boundary. Thus, remembering that the statement of Fourier's law, $q_n = -k\,\frac{\partial T}{\partial n}$, is independent of the actual temperature distribution, and selecting the direction of q_n, conveniently such that it becomes positive, we have from Figure 1.11.

$$\pm k\,\frac{\partial T}{\partial n} = \pm q'' \tag{1-27}$$

where $\partial/\partial n$ denotes differentiation along the normal of tho boundary. The plus and minus signs of the left-hand side of Equation (1.27) correspond to the differentiations along the inward and outward normals, respectively, and the plus and minus signs of the right-hand side correspond to the heat flux from and to the boundary, respectively.

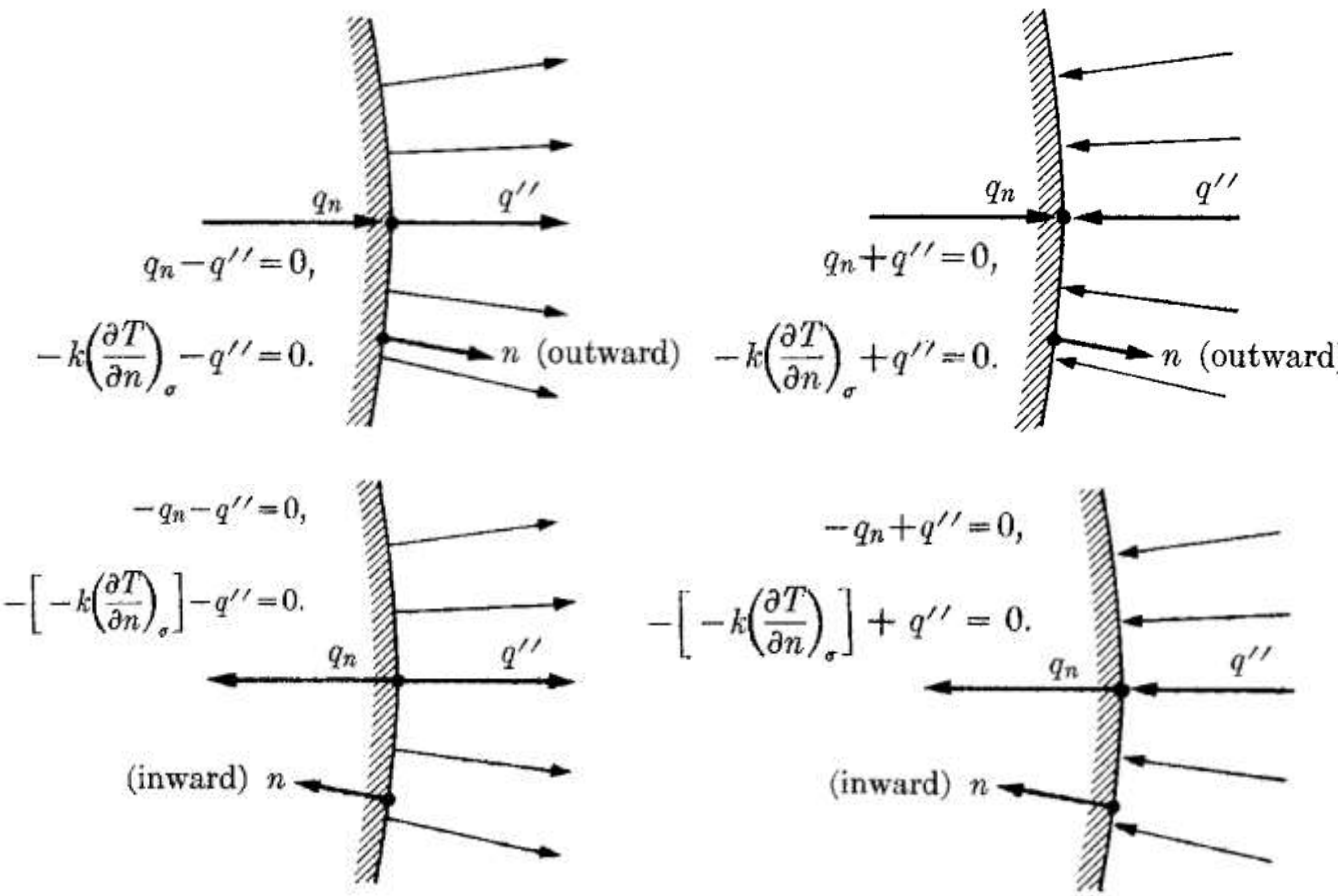


Fig. 1-11: Prescribed surface heat flux boundary conditions.

C. No heat flux (insulation)

This, prescribed is a special form of the previous case, obtained by inserting $q'' = 0$ into Equation (1-27).

$$\frac{\partial T}{\partial n} = 0 \tag{1-28}$$

D. Heat transfer to the ambient by convection

When the heat transfer across the boundaries of a continuum cannot be prescribed, it may be assumed to be proportional to the temperature difference between the boundaries and the ambient. Thus we have

$$q_{conv} = h\,(T - T_\infty) \tag{1-29}$$

where T is the temperature of the solid boundaries, T_∞, is the temperature of the ambient at a distance far from the boundaries, and h, the proportionality constant, is the so-called heat transfer coefficient. Equation (1.29) is Newton's cooling law. The required boundary condition may be stated in the form

$$\pm k\,\frac{\partial T}{\partial n} = h\,(T - T_\infty) \tag{1-30}$$

Where ∂/∂n denotes the differentiation along the normal. The plus and minus signs of the left member of Equation (1.30) correspond to the differentiations along the inward and outward normals, respectively (Figure 1.12). It should be kept in mind that q, shown in Figure 1.12 is a positive quantity, obtained by arbitrarily selecting it in the direction of the normal. Actually, Equation (1.30) is independent of the temperature distribution and the direction of the heat transfer.

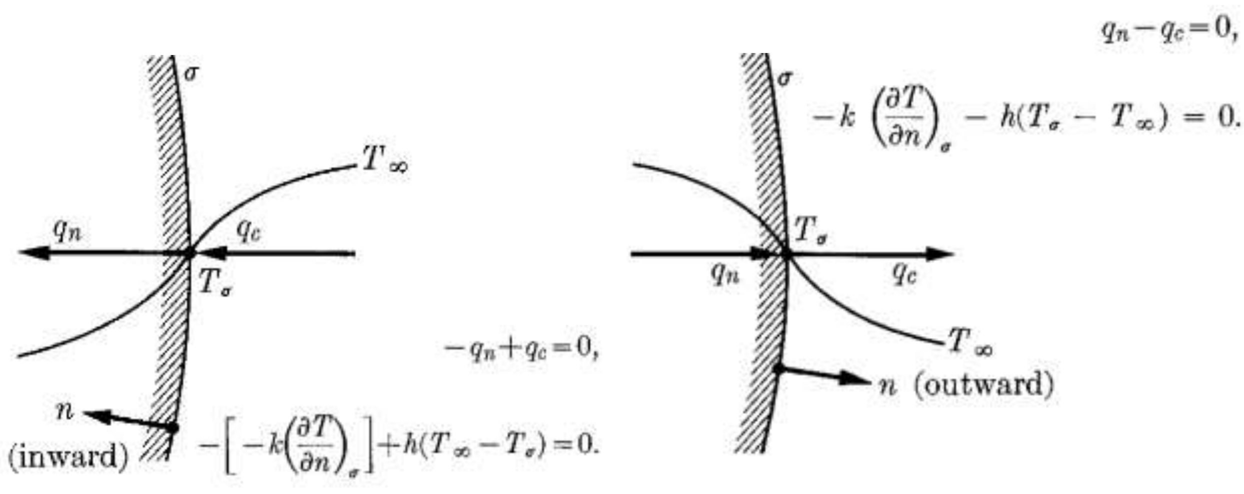


Fig. 1-12: Heat transfer to the ambient by convection surface heat flux boundary conditions.

E. Heat transfer to the ambient by radiation

The boundary condition prescribing heat transfer by radiation from the boundaries of continuum 1. When T_1 is uniform but unspecified, to express the heat flux across the surfaces of 1 by conduction and radiation the required boundary condition may be written in the form

$$\pm k\,\frac{\partial T}{\partial n} = h_{1-2}\sigma\,(T_1^4 - T_2^4) \qquad (1\text{-}31)$$

F. Prescribed heat flux acting at a distance

Consider a continuum that transfers heat to the ambient by convection while receiving the net radiant, heat flux q'' from a distant source (Figure 1.13). The heat transfer coefficient is h, and the ambient temperature T_∞. This boundary condition may be readily obtained as

$$\pm k\,\frac{\partial T}{\partial n} + q'' = h_{1-2}\sigma\,(T_1^4 - T_2^4) \qquad (1\text{-}32)$$

where the signs of the conduction term depend on the direction of normal in the usual manner.

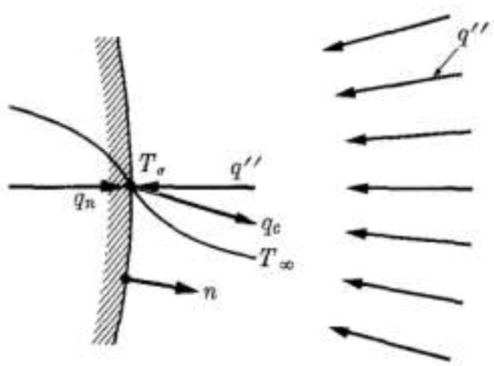


Fig. 1-13: Prescribed heat flux acting at a distance

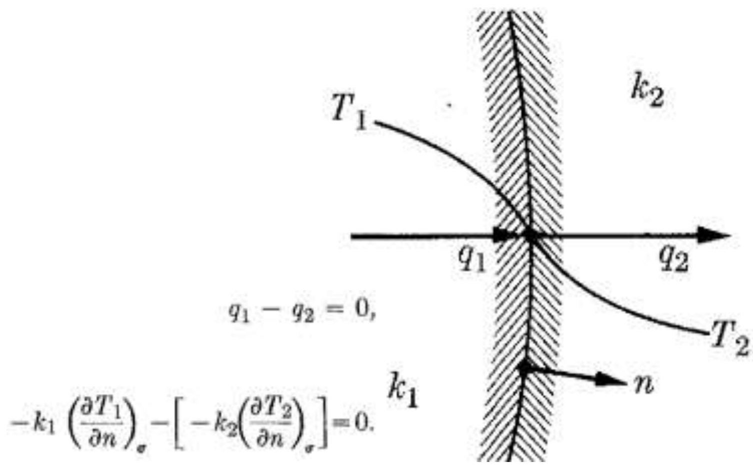


Fig. 1-14: Heat Interface of two continua of different conductivities k_l and k_2

G. Interface of two continua of different conductivities k_l and k_2

When two continua have a common boundary (Figure 1.14), the heat flux across this boundary evaluated from both continua, regardless of the direction of normal, gives

$$k_1 \frac{\partial T_1}{\partial n} = k_2 \frac{\partial T_2}{\partial n} \tag{1-33}$$

H. Interface of two continua in relative motion

Consider two solid continua in contact, one moving relative to the other (Fig. 1.15). The local pressure on the common boundary is *p*, the coefficient of dry friction μ, and the relative velocity *V*. Noting that the heat transfer to both continua by conduction is equal to the work done by friction, we have

$$\pm k_1 \left(\frac{\partial T}{\partial n}\right) + \mu p V = \pm k_2 \frac{\partial T_2}{\partial n} \tag{1-34}$$

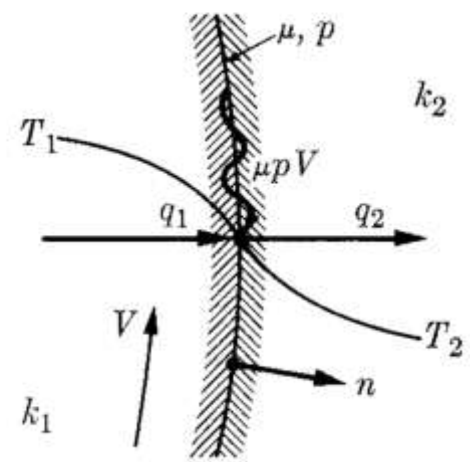


Fig. 1-15: Interface of two continua in relative motion

where the minus signs of the conduction terms correspond to the normal shown in Figure (1.15).

1.4.2 Initial (volume) condition:

For an unsteady problem the temperature of a continuum under consideration must be known at some instant of time. In many cases this instant is most conveniently taken to be the beginning of the problem. Mathematically speaking, if the initial condition is given by $T_o(r)$, the solution of this problem, $T(r, t)$, must be such that at all points of the continuum

$$\lim_{t \to 0} T(r,t) = T_o \tag{1-35}$$

1.5 Methods of investigation and formulation

Four methods are usually used in conduction problems, these are;

1. Analytical Methods
2. Methods of Analogy
3. Computational Methods
4. Graphical Methods

1.5.1 Analytical Methods

In these methods, a number of assumptions are made to simplify the governing equations and get a solution from them. Analytical solution tends to be lengthy and difficult.

1.5.2 Methods of Analogy

A number of lumped distributed models for conduction problems are available based on mechanical, hydrodynamic, and electrical systems. Networks of electrical resistors, capacitors, and sometimes inductors are the most important simulators of lumped systems; on rare occasions, mechanical simulators systems comprised of masses, springs, and dashpots are also used for this purpose. Electrolytic tanks, conductive papers, stretched membranes; soap film, fluid mappers, and polarized light are some of the distributed models occasionally used.

The direct mathematical similarity between heat and electrical conduction is by far the best known and most widely used analogy for the study of complex problems in both steady and transient heat conduction. The characteristic PDE governing the transient distribution of electric potential (electromotive force) E in an electrically-conducting 2-D region of uniform electrical resistance per unit length ($R_L = \frac{R}{L}$) and uniform electrical capacity per unit length ($C_L = \frac{C}{L}$);

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = R_L C_L \frac{\partial T}{\partial t} \quad (1\text{-}36)$$

with the familiar characteristic PDE governing the transient distribution of thermal potential (temperature) T in a thermally conducting 2-D region of uniform diffusivity (α).

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1-37}$$

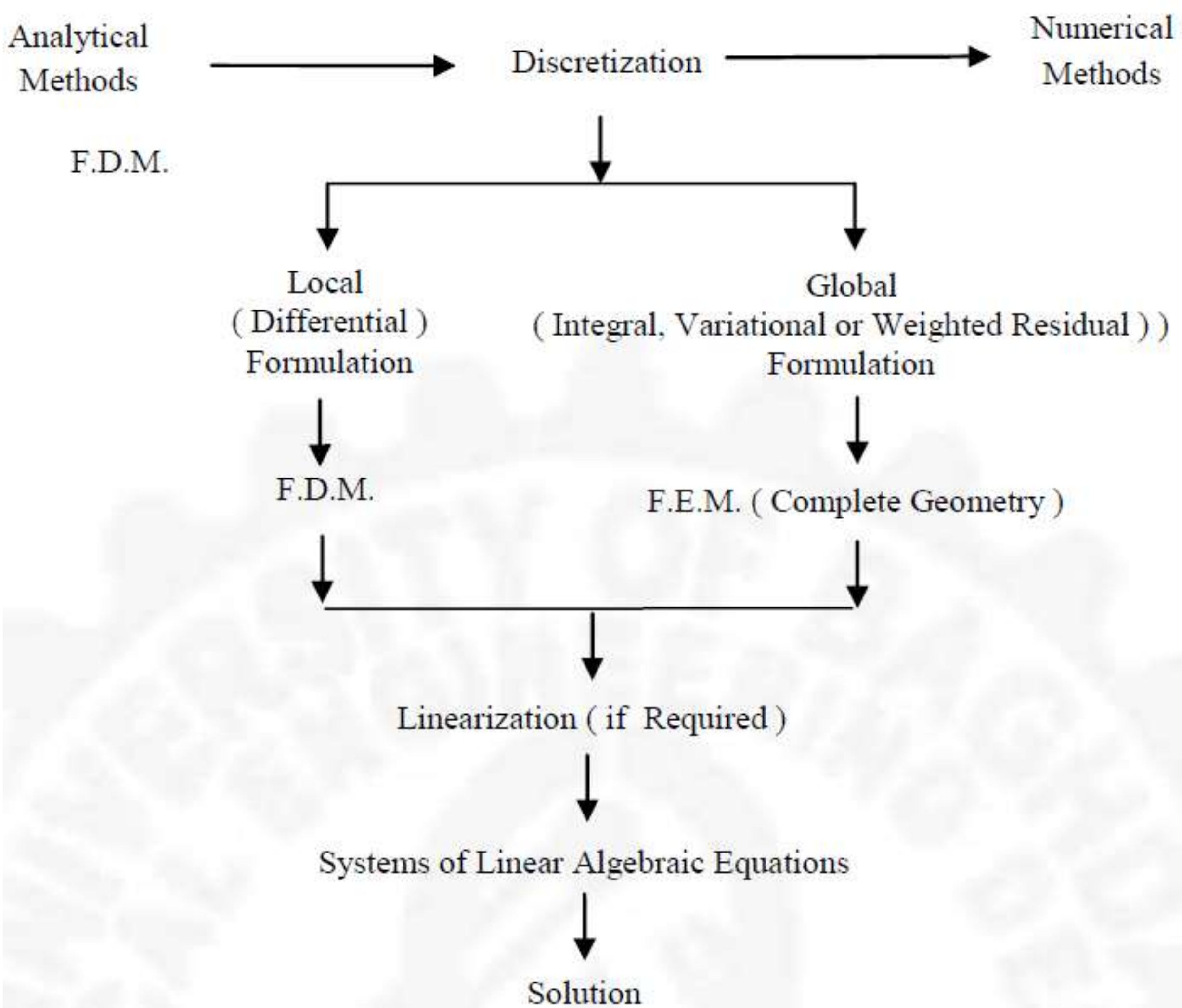
According to previous notations, t represents time. The transient state analogy between electric and temperature potential is therefore complete if on the same time scale the electrical diffusivity ($1/R_L C_L$) and thermal diffusivity (α) are equal. In this state, there is a direct analogy between two laws, the conservation of charge in the electrical system corresponds to the conservation of heat in the thermal system, and the current flow in the electrical system (*Ohm's law*) corresponds to heat flow in the thermal system (*Fourier's law*) . The complete electrical thermal analogy is summarized in Table (1.2).

Table 1.2: Analogues Electrical-Thermal Quantities

Electrical	Thermal
Charge = Q_e(coulomb)	Heat = Q (J , Btu)
Voltage = E (volt)	Temperature = T (C° , F°)
Resistance = R (Ohm)	Resistance R ($\left(\frac{^oC}{W}\right) = \frac{L}{KA} = \frac{1}{hA}$
Current = I (Ampere) = $\frac{\Delta E}{R}$	Flow = q (J/h_r)=$\frac{\Delta T}{R}$
Capacitance= C (Farad) $\frac{I}{\frac{dE}{dt}}$	Unit Capacity = $C_P \rho$V (J/°c) = $\frac{q}{dT/dt}$

1.5.3 Computational Methods

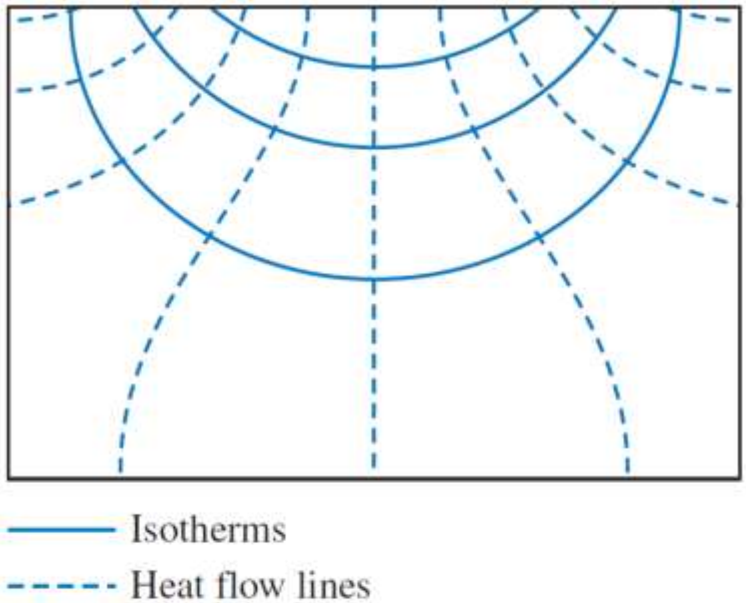
Basically, numerical methods are discretization of analytical methods. By this discretization, the local (differential) formulations leads to a finite difference formulation, while the global (integral, variational, or any other methods of weighted residual) formulation leads to finite element formulation. Both numerical methods lead, after linearization if required, to the solution of systems of linear algebraic equations.



1.5.4 Graphical Methods

The graphic method presented in this section can rapidly yield a reasonably good estimate of the temperature distribution and heat flow in geometrically complex two-dimensional systems, but its application is limited to problems with isothermal and insulated boundaries. The object of a graphic solution is to construct a network consisting of isotherms (lines of constant temperature) and constant-flux lines (lines of constant heat flow). The flux lines are analogous to streamlines in a potential fluid flow, that is, they are tangent to the direction of heat flow at any point. Consequently, no heat can flow across the constant-flux lines. The isotherms are analogous to constant-potential lines, and heat flows perpendicular to them. Thus, lines of constant temperature and lines of constant heat flux intersect at right angles. To obtain the temperature distribution one first prepares a scale model and then draws isotherms and flux lines freehand, by trial and error, until they form a network of curvilinear squares. Then a constant amount of heat flows between any two flux lines. The procedure is illustrated in Figure 1.16 for a corner section of unit depth ($\Delta z = 1$) with faces ABC at temperature T_1, faces FED at temperature T_2, and faces CD and AF insulated. Figure 1.16 (a) shows the scale model, and Figure 1.16 (b) shows the curvilinear network of isotherms and flux lines. It should be noted that the flux lines emanating from isothermal boundaries are perpendicular to the boundary, except when they come from a corner. Flux lines leading to or from a corner of an isothermal boundary bisect the angle between the surfaces forming the corner.

A graphic solution, like an analytic solution of a heat conduction problem described by the Laplace equation and the associated boundary condition, is unique. Therefore, any curvilinear network, irrespective of the size of the squares, that satisfies the boundary conditions represents the correct solution. For any curvilinear square the rate of heat flow is given by Fourier’s law:



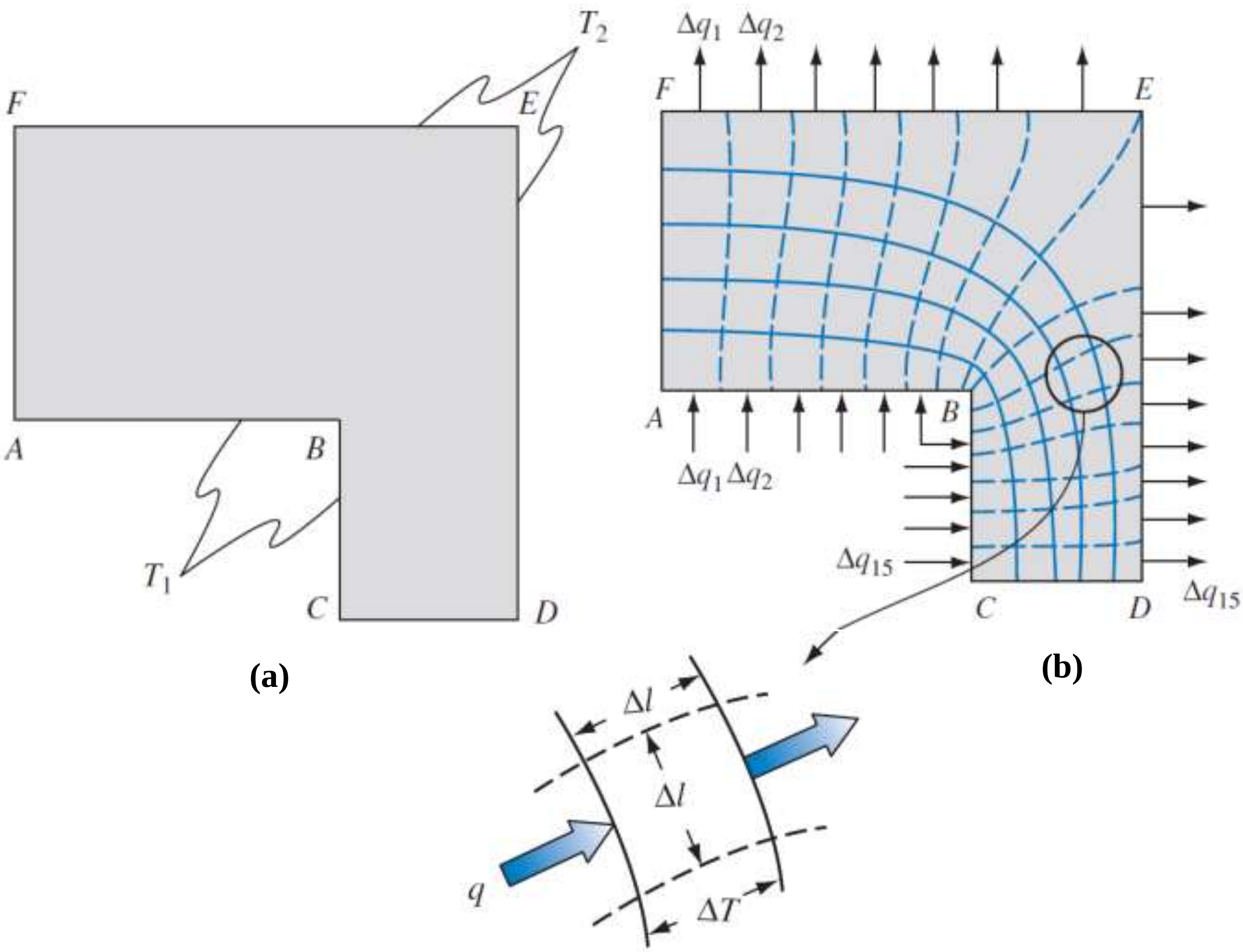


Fig. 1-16: Interface of two continua in relative motion

$$\Delta q = -k(\Delta l \times 1)\frac{\Delta T}{\Delta l} = -k\Delta T \qquad (1\text{-}38)$$

This heat flow will remain the same across any square within any one heat flow lane from the boundary at T_1 to the boundary at T_2. The ΔT across any one element in the heat flow lane is therefore

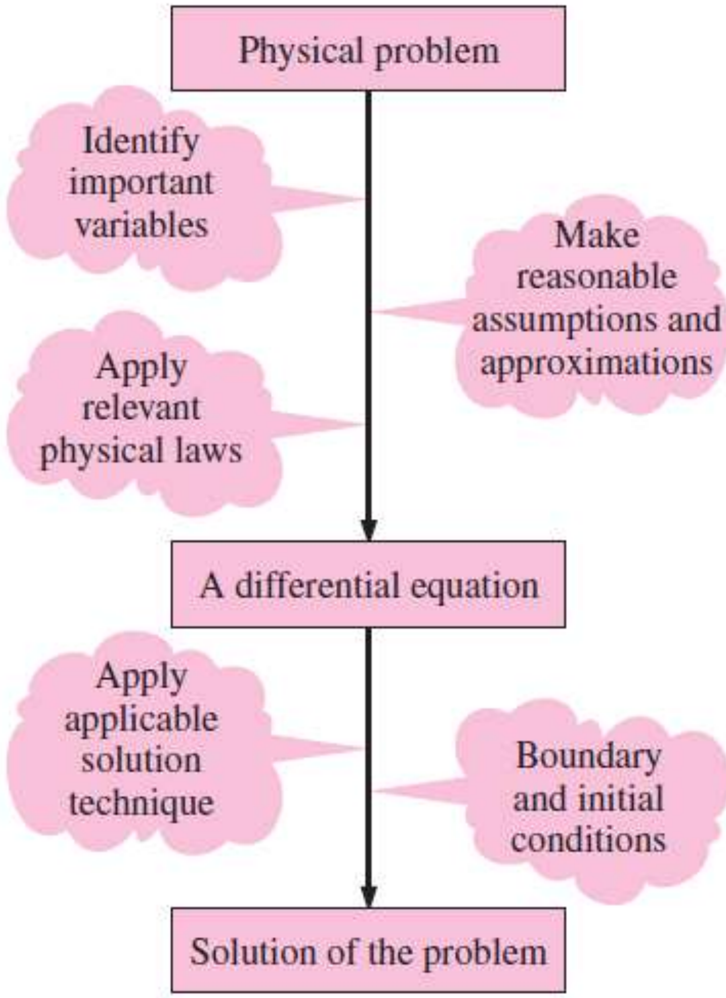
$$\Delta T = \frac{T_2 - T_1}{N} \qquad (1\text{-}39)$$

where N is the number of temperature increments between the two boundaries at T_1 and T_2. The total rate of heat flow from the boundary at T_2 to the boundary at T_1 equals the sum of the heat flow through all the lanes. According to the above relations, the heat flow rate is the same through all lanes since it is independent of the size of the squares in a network of curvilinear squares. The total rate of heat transfer can therefore be written

$$q = \sum_{n=1}^{n=M} \Delta q_n = \frac{M}{N} k(T_2 \times T_1) = \frac{M}{N} k\Delta T_{overall} \qquad (1\text{-}40)$$

where Δq_n is the rate of heat flow through the n^{th} lane, and M is the number of heat flow lanes.

1.6 Mathematical modeling of physical problems



1.7 Homework:

(1) Derive the general three-dimensional conduction heat transfer equation for isotropic heterogeneous medium in cylindrical and spherical coordinates, using energy balance for a finite volume element; obtain the solution for isotropic homogeneous medium ($\alpha = \frac{k}{\rho C}$).

(2) Write down the equation of conduction for the following media in Cartesian coordinates;

a- Heterogeneous anisotropic solids

b- Homogeneous anisotropic solids

c- Heterogeneous isotropic solids

d- Homogeneous isotropic solids

(3) Write down the vectorial and Cartesian forms of the Fourier's law of conduction for heterogeneous anisotropic continua.

(4) What are the most frequently encountered boundary conditions in conduction heat transfer problems? Express these boundary conditions mathematically and mention one application for each boundary condition.

(5) What are the basic modes of heat transfer? And what are the important differences between diffusion and radiation heat transfer?

University of Anbar

College of Engineering

Mechanical Engineering Dept.

Advanced Heat Transfer/ I
Conduction and Radiation

Handout Lectures for MSc. / Power
Chapter Two
One-dimensional, Steady-State Condition

Course Tutor

Assist. Prof. Dr. Waleed M. Abed

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Chapter Two

One-dimensional, Steady-State Condition

2.1 Introduction

Several different physical shapes may fall in the category of one-dimensional systems. Cylindrical and spherical systems are one-dimensional when the temperature in the body is a function only of radial distance and is independent of azimuth angle of axial distance. In some two-dimensional problems the effect of a second – space coordinate may be so small as to justify its neglect, and the multidimensional heat flow problem may be approximated with a one-dimensional analysis. In these cases, the differential equations are simplified, and we are led to a much easier solution as a result of this simplification.

2.2 General Formulation

Consider a long, hollow cylinder or a thick-walled, closed shell of constant wall thickness whose cross section is shown in Figure 2.1. This cylinder or shell contains a fluid at temperature T_i, and is surrounded by an ambient at temperature T_o. Let us suppose that $T_i > T_o$. The inside and outside heat transfer coefficients are h_i and h_o, respectively. We wish to know the temperature distribution of and the heat transfer through this cylinder or shell.

s = space coordinate

$A(s)$ = conduction heat transfer area

$$q_s = q_{s+ds} = q_s + \frac{\partial}{\partial s}(q_s)ds$$

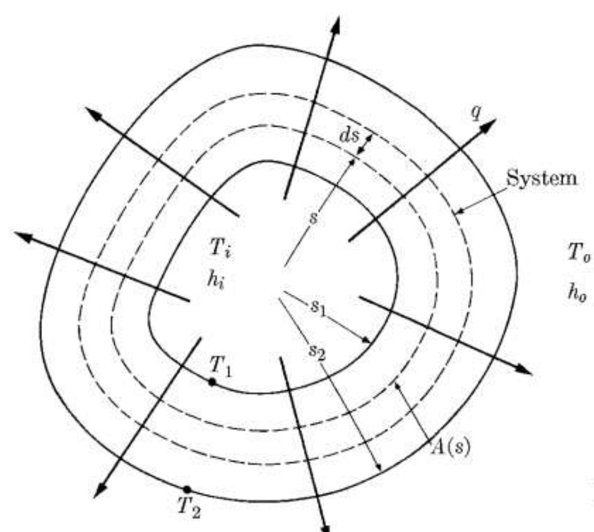


Figure 2.1: a long, hollow cylinder or a thick-walled, closed shell of constant wall thickness.

The method of solution employed here is convenient for one-dimensional problems in which q = *const*. at every cross section.

Assumptions

- ✓ Steady-State Condition
- ✓ One-dimensional problem (length >> thickness)
- ✓ Uniform thermal conductivity (k)
- ✓ Convection heat transfer coefficients (h_i and h_o) are uniform over the whole area.

$$q_s = q_{s+ds} = q_s + \frac{\partial}{\partial s}(q_s)ds \qquad (2\text{-}1)$$

$$q_s = -kA_s \frac{dT}{ds} \qquad (2\text{-}2)$$

$$\frac{d}{ds}\left(-kA_s \frac{dT}{ds}\right) = 0 \qquad (2\text{-}3)$$

From assumption, k = c

$$\frac{d}{ds}\left(A_s \frac{dT}{ds}\right) = 0 \qquad (2\text{-}4)$$

The boundary conditions from Figure 2.1,

Boundary condition (1) $\quad k \frac{dT_{(s1)}}{ds} = h_i[T_{(s1)} - T_i]$

Boundary condition (2) $\quad -k \frac{dT_{(s2)}}{ds} = h_o[T_{(s2)} - T_o]$

By integrating Eq. (2.4) twice, we can get

$$A_s \frac{dT}{ds} = C \quad \rightarrow \quad dT = C \frac{ds}{A_s} \quad \rightarrow \quad \int dT = C \int \frac{ds}{A_s} \quad \rightarrow \quad T = C \int \frac{ds}{A_s} + D$$

Substituting the solution of Eq. (2.4) into both boundary conditions,

$$\frac{kC}{A_{s1}} = h_i[C \int^{s1} \frac{ds}{A_s} + D - T_i] \qquad (2\text{-}5)$$

$$-\frac{kC}{A_{s2}} = h_o[C \int^{s2} \frac{ds}{A_s} + D - T_o] \qquad (2\text{-}6)$$

Thus, after finding C and D from Eqs (2.5 and 2.6) will be,

$$\frac{T - T_o}{T_i - T_o} = U_o[\frac{A_{s2}}{k} \int_s^{s2} \frac{ds}{A_s} + \frac{1}{h_o}] \qquad (2\text{-}7)$$

Where U_o is overall heat transfer coefficient based on A_{s2}

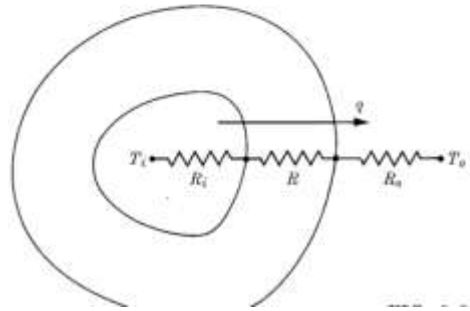
Therefore, $q = \frac{T_i - T_o}{R_i + R + R_o}$ (2-8)

It is sometimes convenient to simplify Eq. (2.8) by writing it in terms of the so-called *over-all coefficient of heat transfer U*, which is defined according to

$q = UA\,(T_i - T_o)$

$\frac{1}{UA} = R_i + R + R_o$

Where, $R_i = \frac{1}{h_i A_{s1}}$, $R_o = \frac{1}{h_o A_{s2}}$, $R = \frac{1}{k}\int_{s1}^{s2} \frac{ds}{A_s}$



Since U depends on A, the statement of U is ambiguous until an area is chosen. Noting that

$UA = U_i A_{s1} = U_o A_{s2}$ (2-9)

where U_i and U_o denote the *over-all heat transfer coefficients* based on the inner and outer surface areas, respectively, we may write the outside coefficient U_o, for example, as

$$\frac{1}{U_o} = \frac{A_{s2}/A_{s1}}{h_i} + \frac{A_{s2}}{k}\int_{s1}^{s2} \frac{ds}{A_s} + \frac{1}{h_o} \qquad (2\text{-}10)$$

For convenience, we shall apply the procedure of the foregoing problem to three important cases, the *Cartesian*, *Cylindrical*, and *Spherical* geometries. The *over-all heat transfer coefficients* based on the outer surface area and the temperature distributions of these geometries are:

Cartesian: $$\frac{1}{U_o} = \frac{1}{U} = \frac{1}{h_i} + \frac{L}{k} + \frac{1}{h_o} \qquad (2\text{-}11)$$

Cylindrical: $$\frac{1}{U_o} = \frac{(R_2/R_1)}{h_i} + \frac{R_2}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_o} \qquad (2\text{-}12)$$

Spherical: $$\frac{1}{U_o} = \frac{(R_2/R_1)^2}{h_i} + \frac{R_2}{k}\left(\frac{R_2}{R_1} - 1\right) + \frac{1}{h_o} \qquad (2\text{-}13)$$

Cartesian: $$\frac{T-T_o}{T_i-T_o} = U_o\left(\frac{x_2-x}{L}+\frac{1}{h_o}\right) \qquad (2\text{-}14)$$

Cylindrical: $$\frac{T-T_o}{T_i-T_o} = U_o\left[\frac{R_2}{k}\ln\left(\frac{R_2}{r}\right)+\frac{1}{h_o}\right] \qquad (2\text{-}15)$$

Spherical: $$\frac{T-T_o}{T_i-T_o} = U_o\left[\frac{R_2}{k}\left(\frac{R_2}{r}-1\right)+\frac{1}{h_o}\right] \qquad (2\text{-}16)$$

Homework:

Derive the following Equations,

Equation 2.7, Equation 2.8, Equation 2.12, Equation 2.3, Equation 2.15 and Equation 2.16.

2.3 Composite Structures

Assume that the hollow cylinder or the thick-walled, closed shell of Figure 2-1 is composed of *N* layers of materials having different thicknesses and thermal conductivities (Figure 2.2). The contact resistance between the layers is negligible. We wish to find the heat transfer from the inner fluid to the surrounding ambient, and the temperature distribution of the structure. Extending the analogy between the diffusion of heat and electric current to the present case, we readily obtain

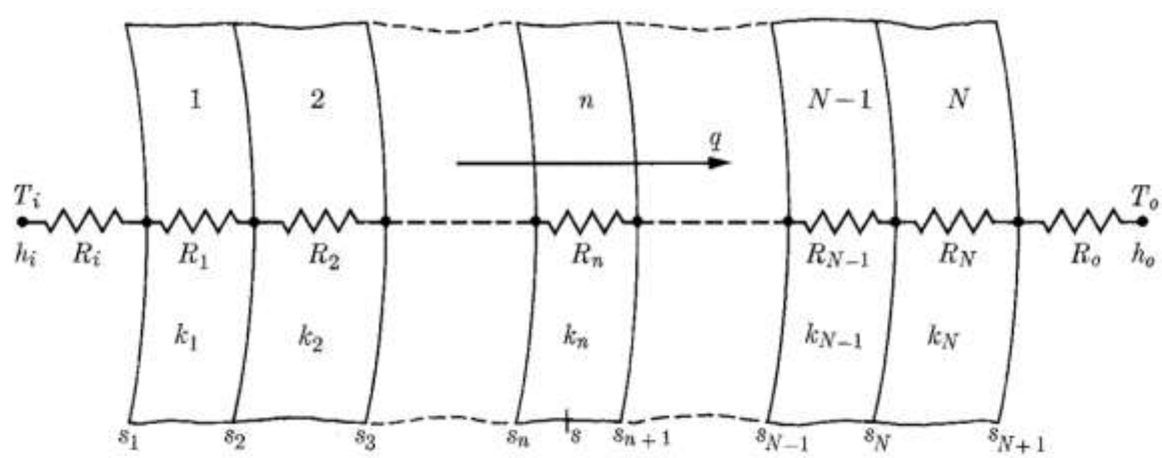


Figure 2.2: The hollow cylinder or the thick-walled with composite wall of *N* layers.

$$\frac{1}{UA} = R_i + \sum_{n=1}^{N} R_n + R_o \tag{2-17}$$

The explicit form of U based on the outside surface is

$$\frac{1}{U_o} = \frac{A_{S_{N+1}}/A_{S_1}}{h_i} + A_{S_{N+1}} \sum_{n=1}^{N} \frac{1}{k_n} \int_{S_n}^{S_{n+1}} \frac{ds}{A_s} + \frac{1}{h_o} \tag{2-18}$$

Equation (2.18) reduces to Eq. (2.10) for $N = 1$.

To obtain the temperature distribution in the structure we first express q in terms of the temperature difference ($T - T_o$) and the corresponding resistances (from the series R_n, R_{n+1}, R_{n+2}, ... ,R_N). The result is

$$q = \frac{T - T_o}{\frac{1}{k_n}\int_{s}^{s_{n+1}}(\frac{ds}{A_s}) + \sum_{m=n+1}^{N}(\frac{1}{k_m})\int_{s_m}^{s_{m+1}}(\frac{ds}{A_s}) + \frac{1}{h_o A_{S_{N+1}}}} \tag{2-19}$$

Where T denotes the temperature of the location s (see Figure 2.2). Then eliminating q between Equation 2.19 and [$q = UA\,(T_i - T_o)$], we find that the desired temperature distribution in terms of U_o is

$$\frac{T - T_o}{T_i - T_o} = U_o[\frac{A_{S_{N+1}}}{k_n} \int_{s}^{s_{n+1}} (\frac{ds}{A_s}) + A_{S_{N+1}} \sum_{m=n+1}^{N} \left(\frac{1}{k_m}\right) \int_{s_m}^{s_{m+1}} \left(\frac{ds}{A_s}\right) + \frac{1}{h_o}] \tag{2-20}$$

Equation (2.20) reduces to Eq. (2.7) for $N = 1$.

The Cartesian, cylindrical, and spherical forms of Equations (2.18) and (2.20) are listed below:

Cartesian: $$\frac{1}{U_o} = \frac{1}{U} = \frac{1}{h_i} + \sum_{n=1}^{N} \frac{L_n}{k_n} + \frac{1}{h_o} \tag{2-21}$$

Cylindrical: $$\frac{1}{U_o} = \frac{(R_{N+1}/R_1)}{h_i} + R_{N+1} \sum_{n=1}^{N} [\frac{1}{k_n} \ln\left(\frac{R_{n+1}}{R_n}\right)] + \frac{1}{h_o} \tag{2-22}$$

Spherical: $$\frac{1}{U_o} = \frac{(R_{N+1}/R_1)^2}{h_i} + R_{N+1}^2 \sum_{n=1}^{N} [\frac{1}{k_n} (\frac{1}{R_n} - \frac{1}{R_{n+1}})] + \frac{1}{h_o} \tag{2-23}$$

Cartesian: $$\frac{T - T_o}{T_i - T_o} = U_o \left(\frac{x_{n+1} - x}{k_n} + \sum_{m=n+1}^{N} (\frac{x_{m+1} - x_m}{k_m}) + \frac{1}{h_o} \right) \tag{2-24}$$

Cylindrical: $$\frac{T - T_o}{T_i - T_o} = U_o \left[\frac{R_{N+1}}{k_n} \ln\left(\frac{R_{n+1}}{r}\right) + R_{N+1} \sum_{m=n+1}^{N} \frac{1}{k_m} \ln(\frac{R_{m+1}}{R_m}) + \frac{1}{h_o} \right] \tag{2-25}$$

Spherical: $\frac{T-T_o}{T_i-T_o} = U_o\left[\frac{R_{N+1}}{k_n}\left(\frac{R_{N+1}}{r}-\frac{R_{N+1}}{R_{n+1}}\right)+R_{N+1}^2\sum_{m=n+1}^{N}\frac{1}{k_m}\left(\frac{1}{R_m}-\frac{1}{R_{m+1}}\right)+\frac{1}{h_o}\right]$

(2-26)

In practice, the combination of series- and parallel-connected structures is also important, especially in Cartesian geometry.

In this section a number of physical and mathematical facts are demonstrated in terms of examples selected from Cartesian, Cylindrical, and Spherical geometries.

Examples:

Q1: Consider a wall composed of hollow concrete blocks, such as is used in building construction (Fig. 2.3). Actually, the heat transfer through this type of wall is not one-dimensional. However, a one-dimensional analysis gives satisfactory results for practical problems.

Solution:

By employing the electrical analogy, we readily obtain

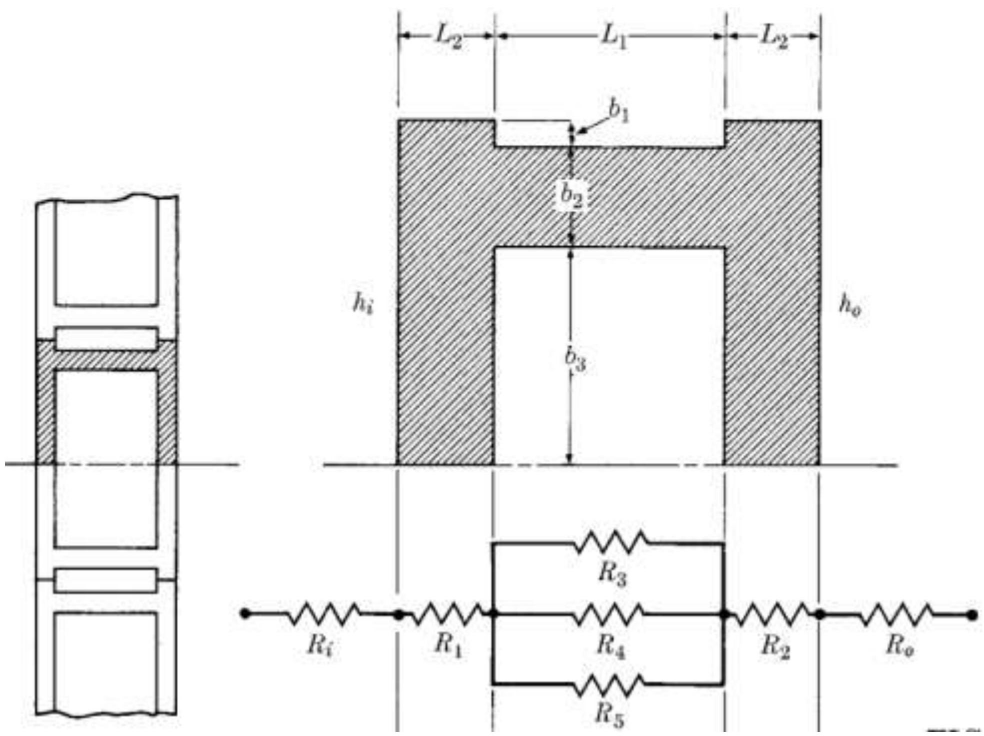


Figure 2.3: A wall composed of hollow concrete blocks.

$\frac{1}{UA} = R_i + \sum_{n=1}^{N} R_n + R_o = R_i + R_1 + \frac{1}{1/R_3 + 1/R_4 + 1/R_5} + R_2 + R_o$

Thus we have, per unit width of the wall perpendicular to the cross section shown in Figure 2.3,

$$\frac{1}{h_i(b_1 + b_2 + b_3)} + \frac{L_2}{k_2(b_1 + b_2 + b_3)} + \frac{1}{k_1 b_1/L_1 + k_2 b_2/L_1 + k_1 b_3/L_1} + \frac{L_2}{k_2(b_1 + b_2 + b_3)} + \frac{1}{h_o(b_1 + b_2 + b_3)},$$

and hence per unit area of the wall

$$\frac{1}{U} = \frac{1}{h_i} + \frac{L_2}{k_2} + \frac{L_1}{\epsilon_1 k_1 + \epsilon_2 k_2} + \frac{L_2}{k_2} + \frac{1}{h_o},$$

where

$$\epsilon_1 = (b_1 + b_3)/(b_1 + b_2 + b_3)$$

and

$$\epsilon_2 = b_2/(b_1 + b_2 + b_3)$$

<u>Q2:</u> The fuel element of a pool reactor is composed of fiat plates of thickness $2L_1$ and cladding material of thickness $(L_2 - L_1)$ bonded to the surfaces of these plates as shown in Figure 2.4. Uniform (nuclear) internal energy $\dot{g}$ is assumed to be generated in the plates only. The heat transfer coefficient is h, the temperature of the coolant T_∞. We need to know the temperature distribution of the fuel element.

<u>Solution:</u>

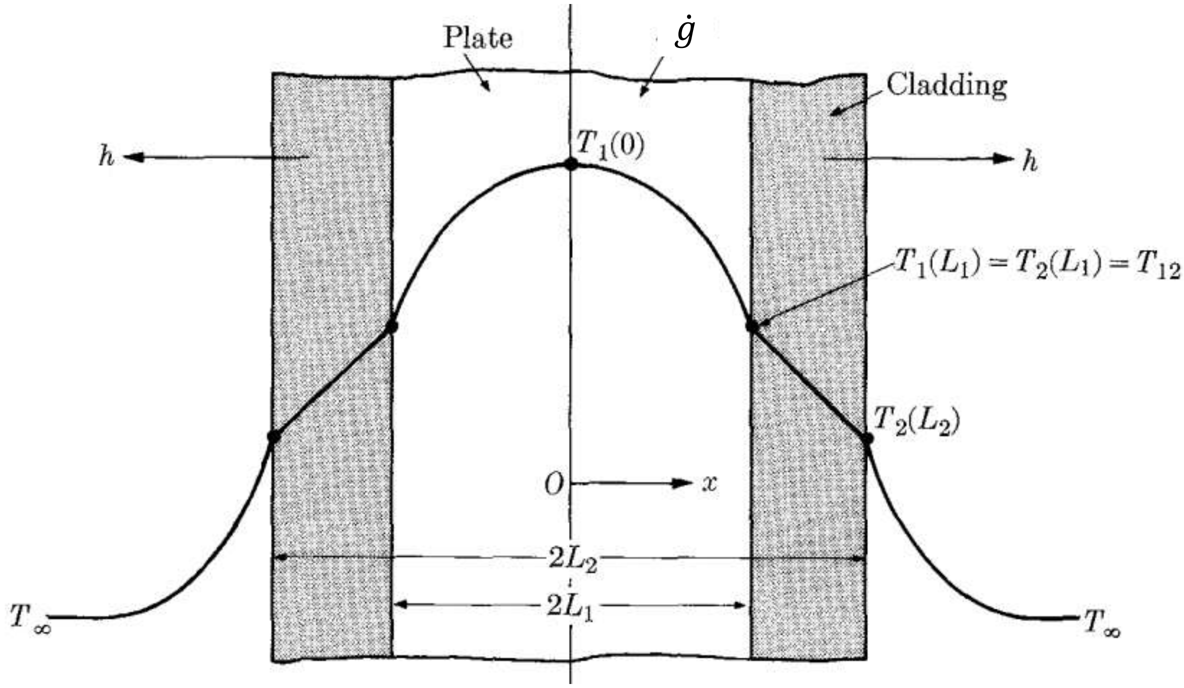
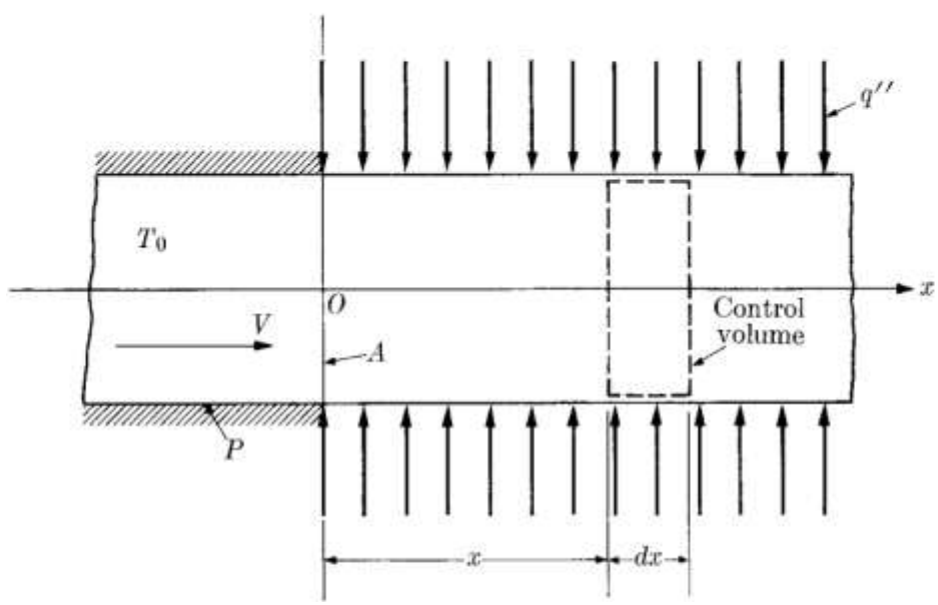


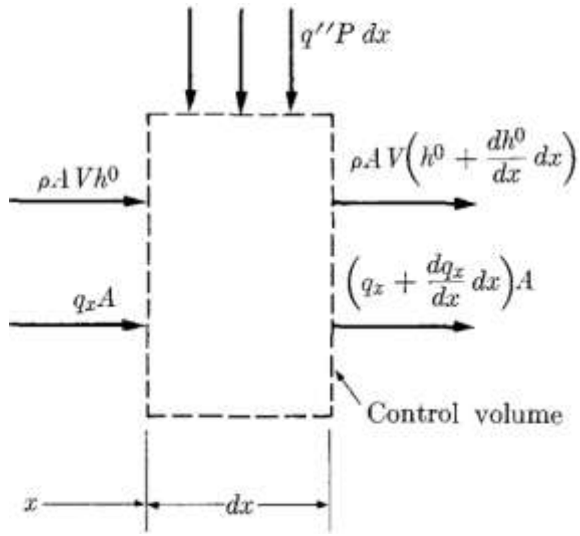
Figure 2.4: The fuel element of a pool reactor.

<u>Q3:</u> A constant-property inviscous liquid having the far upstream temperature T_o and velocity V flows steadily through an infinitely long tube of cross-sectional area A and periphery P. The wall thickness of the tube is negligible. The downstream half of the tube is subjected to the constant heat flux q''; the upstream half is insulated (see Figure 2.5). We wish to know the axial temperature distribution of the liquid, based on a radially lumped analysis.

<u>Solution:</u>



Consider the radially lumped, axially differential control volume shown in Figure 2.5. Let us apply the general laws to this control volume as follows.



Q4: An electric wire of radius R is uniformly insulated with plastic to produce an outer radius R_o (see Figure 2.6). The electrical resistance and thermal conductivity of this wire are ρ_e (*ohms* × *length*) and k_w, respectively. The thermal conductivity of the insulation is k, the heat transfer coefficient h, and the ambient temperature T_∞. We wish to determine the maximum current that this wire can carry without heating the plastic above its allowable operating temperature T_{max}.

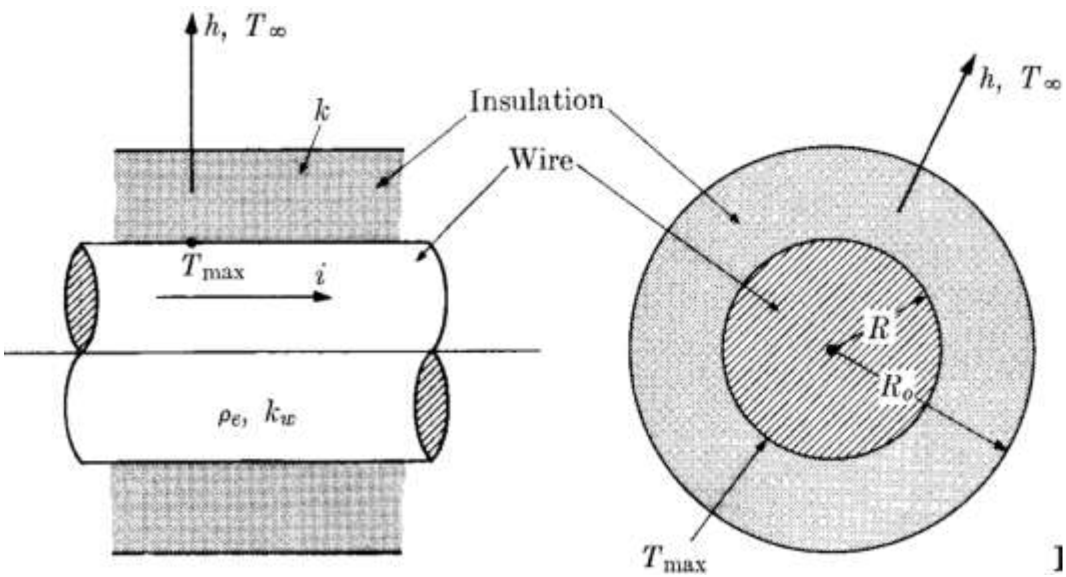


Figure 2.6: An electric wire.

Q5: The fuel element of a reactor consists of a sphere of fissionable material with radius R, surrounded by a spherical shell of cladding with outer radius R_o (see Figure 2.7). The temperature of the coolant is T_∞, and the heat transfer coefficient is h. The nuclear internal energy generated in the sphere can be approximated by a parabola as: $\dot{g}_{(r)} = \dot{g}_o[1 - \left(\frac{r}{R}\right)^2]$, where $\dot{g}_o$ is the nuclear energy generation at the center of the sphere. We wish to know the temperature distribution in the fuel element.

Solution:

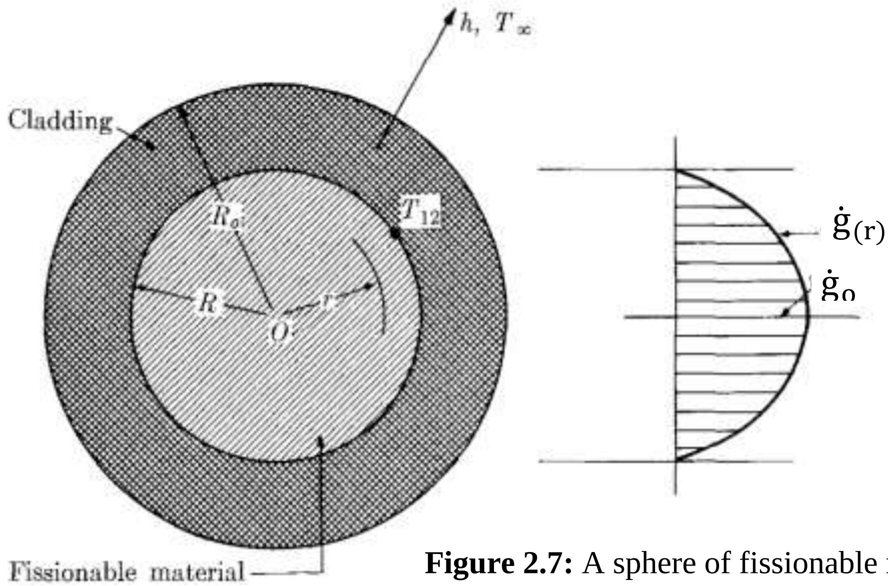


Figure 2.7: A sphere of fissionable material.

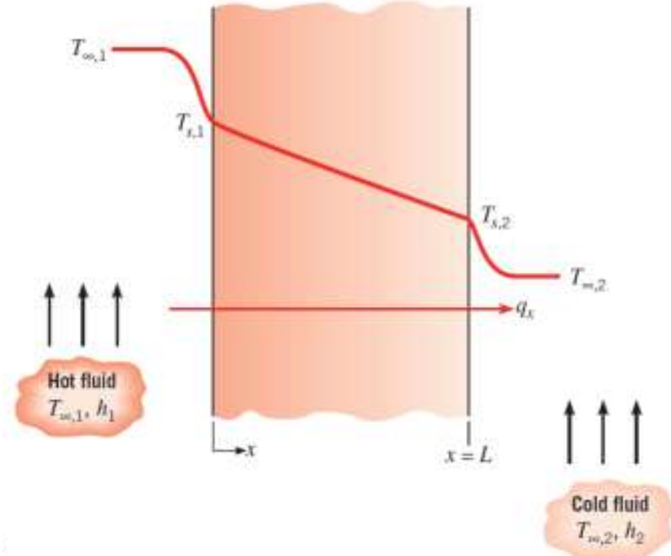
Q6: Derive an expression for the temperature distribution and the heat flow through the walls of the following three geometries for the following four cases (each of cases (**i** & **ii**) should be worked with both cases (**iii** & **iv**)):

Case (i): without heat generation, **Case (ii):** with heat generation ($\dot{g}$)

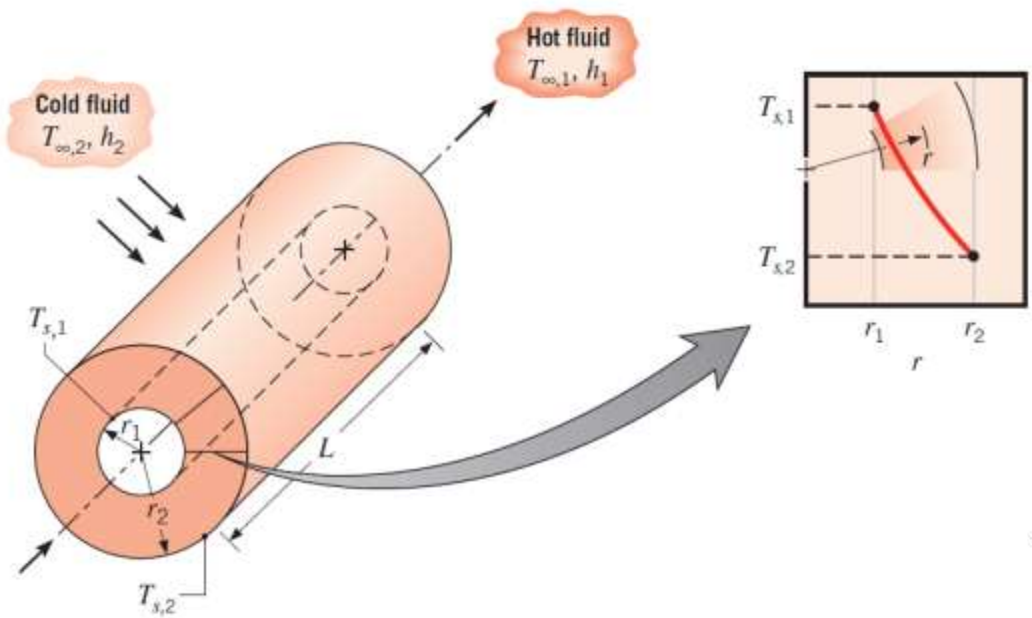
Case (iii): constant thermal conductivity (k).

Case (iv): variable thermal conductivity ($k = k_o(1 + \beta\, T)$), where (k_o) is known conductivity at a reference temperature (T_o) and (β) is the coefficient of thermal conductivity.

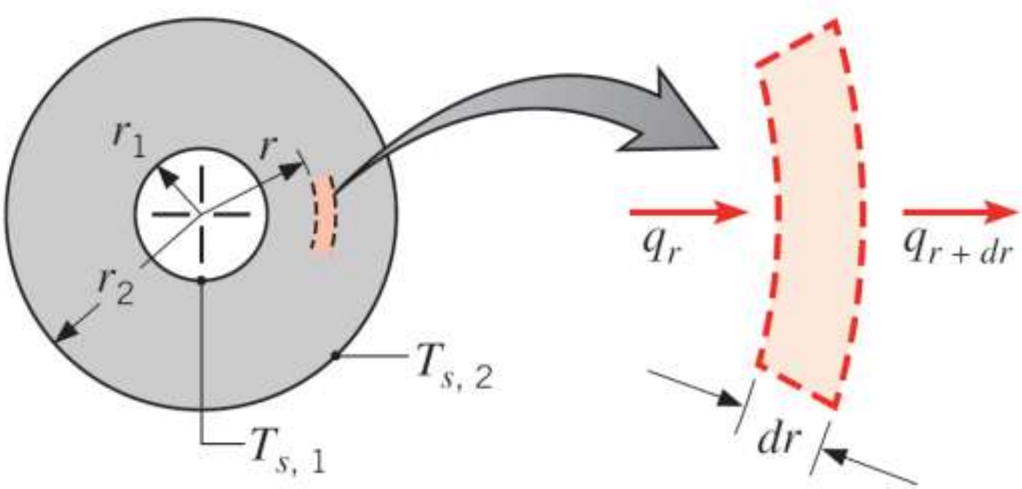
Geometry (I) In finite plate wall



Geometry (II) Hollow long cylinder



Geometry (III) Hollow sphere



University of Anbar

College of Engineering

Mechanical Engineering Dept.

Advanced Heat Transfer/ I
Conduction and Radiation

Handout Lectures for MSc. / Power
Chapter Three
Heat Transfer from Extended Surfaces

Course Tutor

Assist. Prof. Dr. Waleed M. Abed

- J. P. Holman, "*Heat Transfer*", McGraw-Hill Book Company, 6th Edition, 2006.
- T. L. Bergman, A. Lavine, F. Incropera, D. Dewitt, "*Fundamentals of Heat and Mass Transfer*", John Wiley & Sons, Inc., 7th Edition, 2007.
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Chapter Three
Heat Transfer from Extended Surfaces

3.1 Introduction

The term "*extended surface*" is commonly used to depict an important special case involving heat transfer by conduction within a solid and heat transfer by convection (and/or radiation) from the boundaries of the solid.

Consider first a wall at temperature T_w transferring heat by convection to an ambient at temperature T_∞. Therefore, the rate of heat transfer from this wall may be evaluated in terms of a heat transfer coefficient in the form,

$$q_{conv.} = hA(T_w - T_\infty) \qquad (3\text{-}1)$$

Clearly, $q_{conv.}$ of Eq. (3.1) may be increased by increasing:

(***i***) The temperature difference between the wall and the ambient.

(***ii***) The heat transfer coefficient.

(***iii***) The heat transfer area.

The first case needs no explanation; the second case is the subject matter of texts on convective heat transfer; the third case is the concern of this section.

Examples of extended surfaces (fins) applications are easy to find around us. Consider the arrangement for cooling engine heads on motorcycles and lawn mowers or for cooling electric power transformers (see Figure 3.1). Consider also the tubes with attached fins used to promote heat exchange between air and the working fluid of an air conditioner. Two common finned-tube arrangements are shown in Figure 3.1.

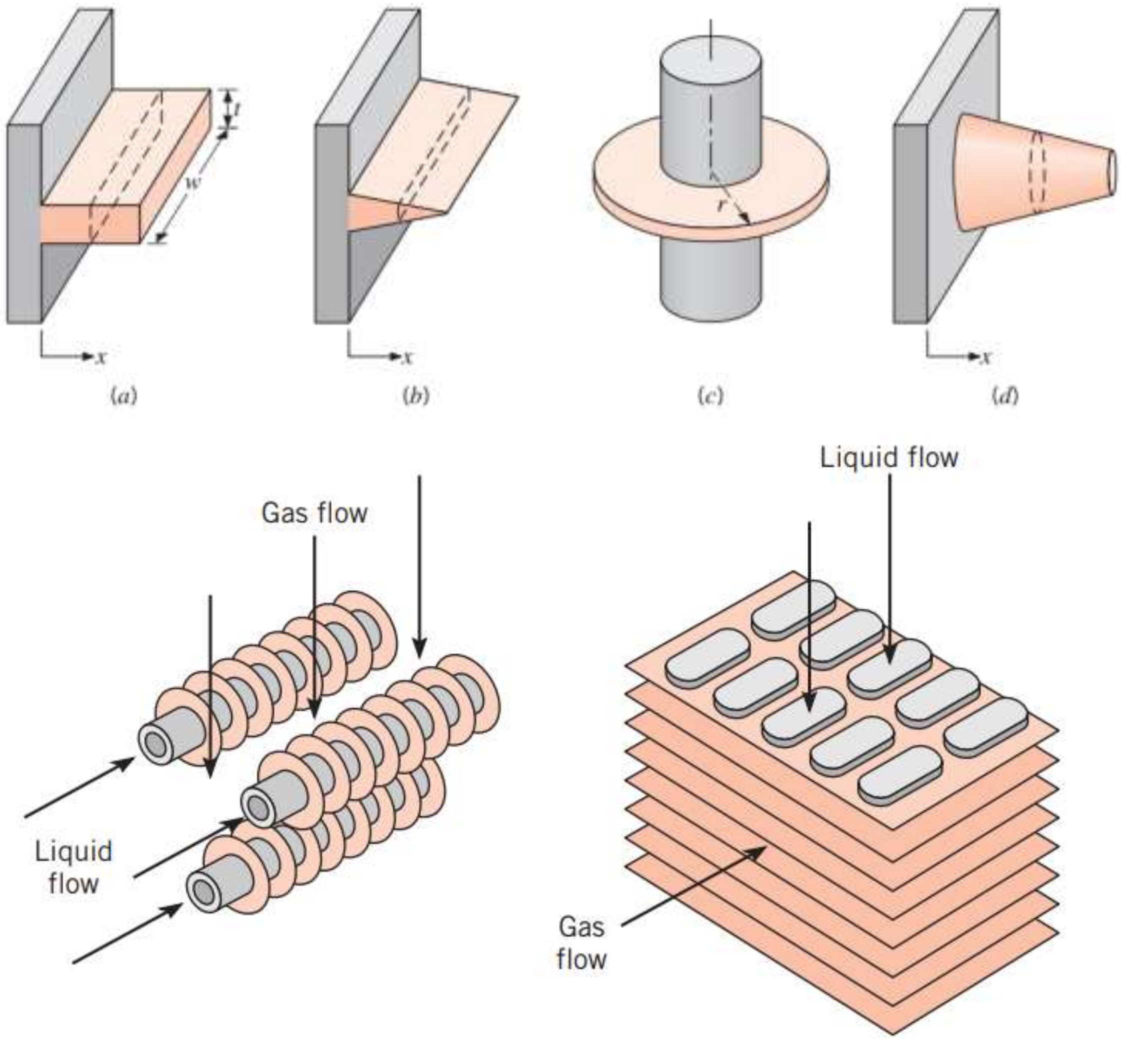


Figure 3.1: Fin configurations. (a) Straight fin of uniform cross section. (b) Straight fin of non-uniform cross section. (c) Annular fin. (d) Pin fin, and Schematic of typical finned-tube heat exchangers.

3.2 Analysis of extended surfaces (fins)

Consider the extended surface of Figure 3.2. The analysis is simplified if certain assumptions are made. We choose to assume one-dimensional conditions in the longitudinal (x-) direction, even though conduction within the fin is actually two-dimensional. The rate at which energy is convected to the fluid from any point on the fin surface must be balanced by the net rate at which energy reaches that point due to conduction in the transverse (y-, z-) direction. However, in practice the fin is thin, and temperature changes in the transverse direction within the fin are small compared with the temperature difference between the fin and the environment. Hence, we may assume that the temperature is uniform across the fin thickness, that is, it is only a function of *x*.

We will consider steady-state conditions and also assume that the thermal conductivity is constant, that radiation from the surface is negligible, that heat generation effects are absent, and that the convection heat transfer coefficient *h* is uniform over the surface. Steady state (T , $q \neq f(t)$). Homogeneous material. Uniform free stream temperature (T_∞). Uniform base temperature. Negligible contact resistance. No heat generation. By applying the conservation of energy requirement,

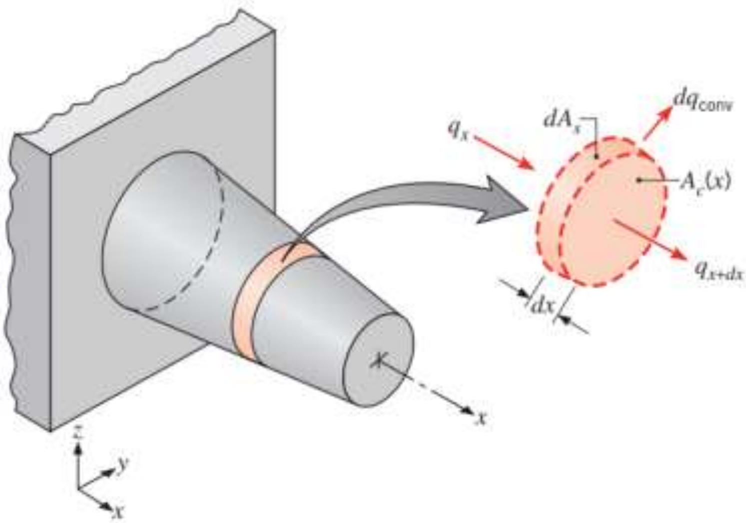


Figure 3.2: Energy balance for an extended surface.

$$q_x = q_{x+dx} + dq_{conv.} \tag{3-2}$$

$$-kA_c\frac{dT}{dx} = -kA_c\frac{dT}{dx} - k\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)dx + hdA_s(T - T_\infty) \tag{3-3}$$

where A_c is the cross-sectional area, which may vary with *x*. dA_s is the surface area of the differential element.

$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{hdA_s}{k\,dx}(T - T_\infty) = 0 \tag{3-4}$$

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{hdA_s}{k\,dx}\right)(T - T_\infty) = 0 \tag{3-5}$$

This result provides a general form of the energy equation for an extended surface. Its solution for appropriate boundary conditions provides the temperature distribution, which may be used to calculate the conduction rate at any *x*.

3.2.1 Extended surfaces with constant cross sections

To solve Equation 3.5 it is necessary to be more specific about the geometry. We begin with constant (uniform) cross-sectional area such as the simplest case of straight rectangular and pin fins (see Figure 3.1 and Figure 3.3). Each fin is attached to a base surface of temperature $T_{(x=0)} = T_b$ and extends into a fluid of temperature T_∞.

For the prescribed fins, A_c is a constant and $A_s = Px$, where A_s is the surface area measured from the base to *x* and *P* is the fin perimeter. Accordingly, with $dA_c/dx = 0$ and $dA_s/dx = P$, Equation 3.5 reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{k\,A_c}(T - T_\infty) = 0 \qquad (3\text{-}6)$$

To simplify the form of this equation, we transform the dependent variable by defining an excess temperature θ as,

$$\theta_{(x)} = T_{(x)} - T_\infty \qquad (3\text{-}7)$$

where, since T_∞ is a constant, $d\theta/dx = dT/dx$, and $m^2 = (hP/kA_c)$, Substituting Equation 3.7 into Equation 3.6, we then obtain

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (3\text{-}8)$$

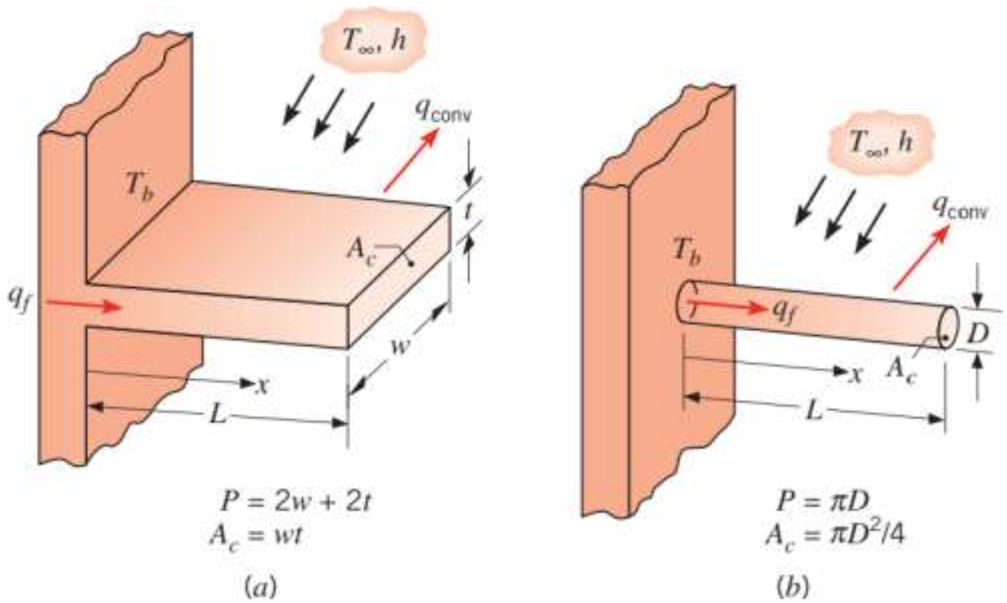


Figure 3.3: Straight fins of uniform cross section. (a) Rectangular fin. (b) Pin fin.

Equation 3.8 is a linear, homogeneous, second-order differential equation with constant coefficients. Its general solution is of the form

$$\theta_{(x)} = C_1 e^{mx} + C_1 e^{-mx} \qquad (3\text{-}9)$$

Or, $$\theta_{(x)} = C_3 \cosh mx + C_4 \sinh mx \qquad (3\text{-}10)$$

To evaluate the constants C_1 and C_2 of Equation 3.9 and C_3 and C_4 of Equation 3.10, it is necessary to specify appropriate boundary conditions. One such condition may be specified in terms of the temperature at the ***base*** of the fin (x= 0),

$$\theta_{(x=0)} = \theta_b = T_b - T_\infty \qquad (3\text{-}11)$$

The second condition, specified at the fin ***tip*** (x = L), may correspond to one of four different physical situations.

<u>**Case A**</u>: considers *convection heat transfer from the fin tip*. Applying an energy balance to a control surface about this tip (Figure 3.4), we obtain

$$hA_c\left(T_{(x=L)} - T_\infty\right) = -kA_c \left.\frac{dT}{dx}\right|_{x=L} \qquad \text{or} \qquad h\theta_{(L)} = -k \left.\frac{d\theta}{dx}\right|_{x=L}$$

Solving for C_1 and C_2, it may be shown, after some manipulation, that

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

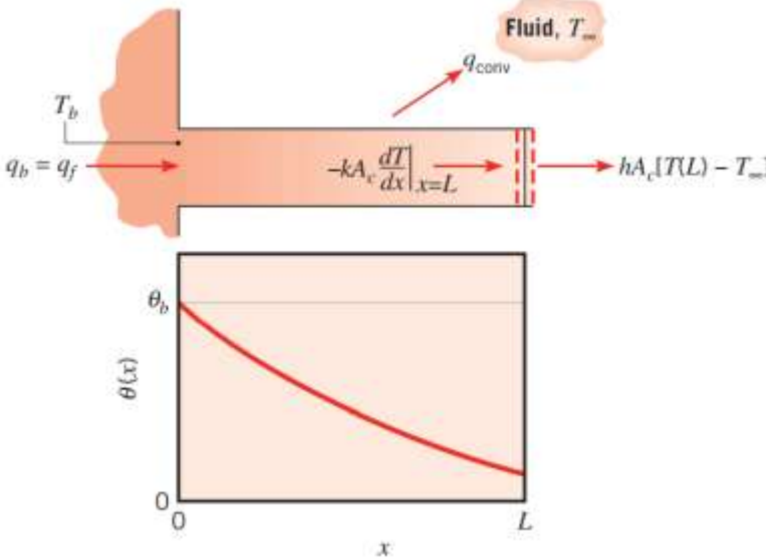


Figure 3.4: Conduction and convection in a fin of uniform cross section.

The form of this temperature distribution is shown schematically in Figure 3.4. Note that the magnitude of the temperature gradient decreases with increasing x. This trend is a consequence of the reduction in the conduction heat transfer $q_x(x)$ with increasing x due to continuous convection losses from the fin surface. We are particularly interested in the amount of heat transferred from the entire fin. From Figure 3.4, it is evident that the fin heat transfer rate q_f may be evaluated in two alternative ways, both of which involve use of the temperature distribution. The simpler procedure, and the one that we will use, involves applying *Fourier's law* at the fin base. That is,

$$q_f = q_b = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c \frac{d\theta}{dx}\Big|_{x=0} \qquad (3\text{-}13)$$

Hence, knowing the temperature distribution, $\theta(x)$, q_f may be evaluated, giving

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \qquad (3\text{-}14)$$

Case B: corresponds to the assumption that the convective heat loss from the fin tip is negligible, in which case *the tip may be treated as adiabatic* and

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

Using this boundary condition to solve for C_1 and C_2 and substituting the results into Equation 3.9, we obtain

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL} \quad (3\text{-}15)$$

Using this temperature distribution with Equation 3.13, the fin heat transfer rate is then

$$q_f = \sqrt{hPkA_c}\theta_b \tanh mL \quad (3\text{-}16)$$

<u>Case C:</u> where *the temperature is prescribed at the fin tip*. That is, the second boundary condition is $\theta_{(L)} = \theta_L$, and the resulting expressions are of the form

$$\frac{\theta}{\theta_b} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL} \quad (3\text{-}17)$$

Using this temperature distribution with Equation 3.13, the fin heat transfer rate is then

$$q_f = \sqrt{hPkA_c}\theta_b \ \frac{\cosh mL - (\theta_L/\theta_b)}{\sinh mL} \quad (3\text{-}18)$$

<u>Case D:</u> is an interesting extension of these results. In particular, as *Infinite fin* ($L \rightarrow \infty$, $\theta_{(L)} \rightarrow 0$) and it is easily verified that

$$\frac{\theta}{\theta_b} = e^{-mx} \quad (3\text{-}19)$$

$$q_f = \sqrt{hPkA_c}\theta_b \quad (3\text{-}20)$$

<u>Homework 1:</u>

Prove the following Equations,

Equation 3.12, Equation 3.14, Equation 3.15, Equation 3.16, Equation 3.17, Equation 3.18, Equation 3.19, Equation 3.20.

Examples:

Q1: A metal rod of length 2L, diameter D, and thermal conductivity k is inserted into a perfectly insulating wall, exposing one-half of its length to an airstream that is of temperature T_∞ and provides a convection coefficient h at the surface of the rod as shown in Figure 3.5. An electromagnetic field induces volumetric energy generation at a uniform rate $\dot{g}$ within the embedded portion of the rod.

(**a**) Derive an expression for the steady-state temperature T_b at the base of the exposed half of the rod. The exposed region may be approximated as a very long fin.

(**b**) Derive an expression for the steady-state temperature T_o at the end of the embedded half of the rod.

(**c**) Using numerical values provided in the schematic, plot the temperature distribution in the rod and describe key features of the distribution. Does the rod behave as a very long fin?

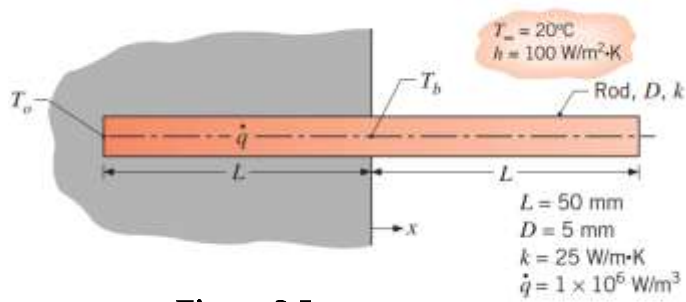


Figure 3.5

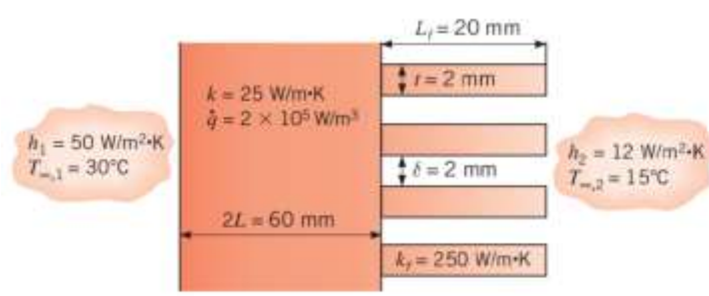


Figure 3.6

Q2: Heat is uniformly generated at the rate of 2×10^5 W/m^3 in a wall of thermal conductivity 25 W/m K and thickness 60 mm. The wall is exposed to convection on both sides, with different heat transfer coefficients and temperatures as shown in Figure 3.6. There are straight rectangular fins on the right-hand side of the wall, with dimensions as shown and thermal conductivity of 250 W/mK. What is the maximum temperature that will occur in the wall?

Q3: A constant-area fin between surfaces at temperatures T_1 and T_2 is shown in Figure 3.7. If the external temperature, $T_\infty(x)$, is a function of the coordinate x, find the general steady-state solution of the fin temperature $T(x)$ for (**a**) $T_1 = T_2$ and (**b**) $T_1 \neq T_2$.

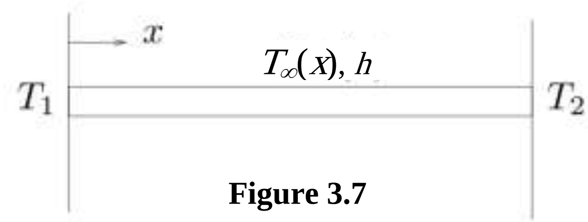


Figure 3.7

3.2.2 Bessel Functions

In section 3.2.3, a class of one-dimensional problems associated with extended surfaces (fins, pins, or spines) will be discussed. When the cross section of an extended surface is variable, the formulation of the problem results in a *second-order* linear differential equation with variable coefficients. This differential equation is a form of *Bessel's equation*, except in a special case which leads to the so-called *equidimensional equation*. The solution methods suitable to *second-order* linear differential equations with constant coefficients are not suitable to those with variable coefficients. We may, however, recall that equations with variable coefficients possess solutions expressible, over an appropriate interval, in terms of power series. This section is therefore devoted to a brief review of the power series solution of *Bessel's equation*.

The general *Bessel's equation* is,

$$X^2\frac{d^2y}{dx^2} + [(1-2A)X - 2BX^2]\frac{dy}{dx} + [C^2D^2X^{2C} + B^2X^2 - B(1-2A)X + A^2 - C^2n^2]y = 0$$

The solution of *Bessel's equation*:

$$y = X^A e^{BX}[C_1 J_n(DX^C) + C_2 Y_n(DX^C)] \qquad \text{(when } D \text{ is } real\text{)}$$

$$y = X^A e^{BX}[C_1 I_n(DX^C) + C_2 K_n(DX^C)] \qquad \text{(when } D \text{ is } imaginary\text{)}$$

Where:

J_n : Ordinary *Bessel function* of first kind of order *n*.

Y_n : Ordinary *Bessel function* of second kind of order *n*.

I_n : Modified *Bessel function* of first kind of order *n*.

K_n : Modified *Bessel function* of second kind of order *n*.

Ordinary Bessel Function:

$J_o(x), J_1(x)$: Ordinary *Bessel function* of first kind of order, zero and first order respectively.

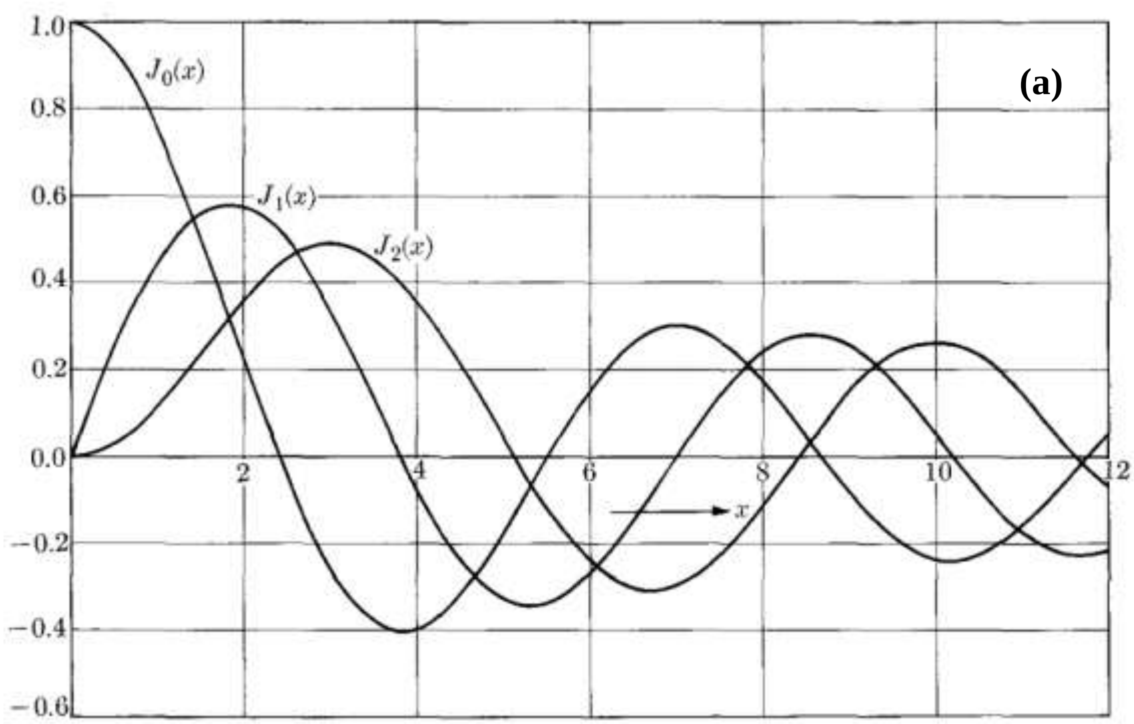
$Y_o(x), Y_1(x)$: Ordinary *Bessel function* of second kind of order, zero and first order respectively.

Modified Bessel function:

$I_o(x), I_1(x)$: *Modified Bessel function* of first kind of order, zero and first order respectively.

$K_o(x), K_1(x)$: *Modified Bessel function* of second kind of order, zero and first order respectively.

Graphical representation of the general behavior of *Bessel functions*. Graphs of the general behavior of *Bessel functions* are shown in Figure 3.8. Having thus completed our review of *Bessel functions* we may now proceed to demonstrate the use of these functions in the solution of problems related to extended surfaces.



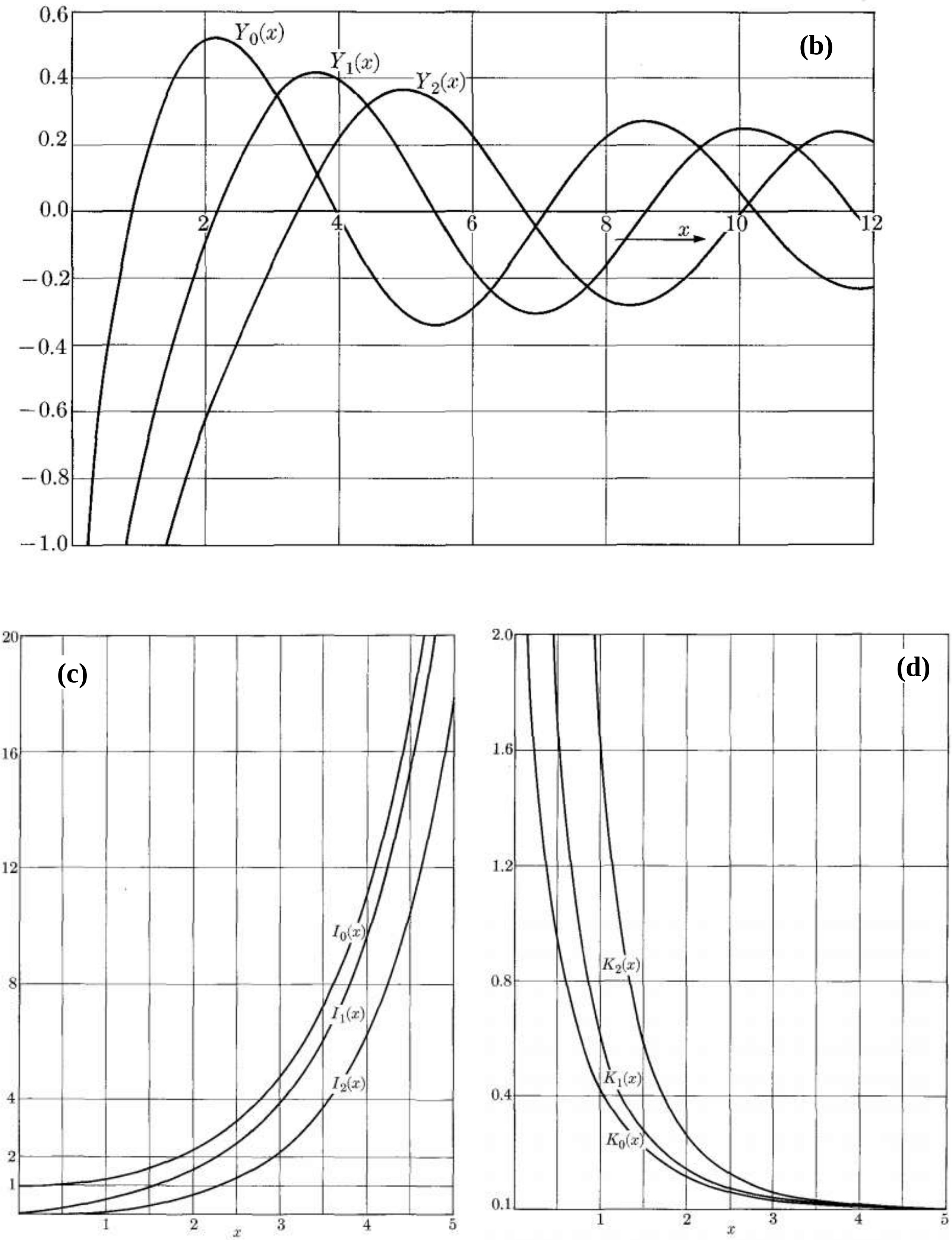

(b)
$Y_0(x)$
$Y_1(x)$
$Y_2(x)$
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
2
4
6
8
10
12
x
(c)
$I_0(x)$
$I_1(x)$
$I_2(x)$
20
16
12
8
4
2
1
1
2
3
4
5
x
(d)
$K_2(x)$
$K_1(x)$
$K_0(x)$
2.0
1.6
1.2
0.8
0.4
0.1
1
2
3
4
5
x

Figure 3.8

Example-1:

$$X^2\frac{d^2y}{dx^2}+X\frac{dy}{dx}+\lambda^2X^2y=0$$

Solution:

The comparison between the above equation and *Bessel functions* yields to,

$$X^2\frac{d^2y}{dx^2}+[(1-2A)X-2BX^2]\frac{dy}{dx}+[C^2D^2X^{2C}+B^2X^2-B(1-2A)X+A^2-C^2n^2]y=0$$

Therefore,

$1-2A=1 \xrightarrow{yields} A=0$

$2B=0 \xrightarrow{yields} B=0$

$2C=2 \xrightarrow{yields} C=1$

$C^2D^2=\lambda^2 \xrightarrow{yields} D=\lambda$

$C^2n^2=0 \xrightarrow{yields} n=0$

$y_{(x)}=C_1J_0(\lambda X)+C_2Y_0(\lambda X)$

Example-2:

$y''+\frac{1}{X}y'-\mu^2y=0 \qquad (\times X^2)$

$X^2y''+Xy'-\mu^2X^2y=0$

Solution:

The comparison between the above equation and *Bessel functions* yields to,

$$X^2\frac{d^2y}{dx^2}+[(1-2A)X-2BX^2]\frac{dy}{dx}+[C^2D^2X^{2C}+B^2X^2-B(1-2A)X+A^2-C^2n^2]y=0$$

Therefore,

$1-2A=1 \xrightarrow{yields} A=0$

$2B=0 \xrightarrow{yields} B=0$

$2C=2 \xrightarrow{yields} C=1$

$C^2D^2=-\mu^2 \xrightarrow{yields} D=i\mu$

$C^2n^2=0 \xrightarrow{yields} n=0$

$y_{(x)}=C_1I_0(\mu X)+C_2K_0(\mu X)$

3.2.3 Extended surfaces with variable cross sections

The general formulation of problems of extended surfaces with variable cross sections has already been given by Eq. (3.4) [$\frac{d}{dx}\left(A_c\frac{d\theta}{dx}\right)+\frac{hdA_s}{k\,dx}\theta=0$]. Since A_c and A_s (Pdx) are no longer constant, this equation now becomes a differential equation with variable coefficients whose general solution can be determined only when A_c and A_s are specified. In most cases Eq. (3.4) is reduced to a form of *Bessel's equation*; a special case is that leading to the *equidimensional equation*. Cases which do not lead to either of these equations may be treated individually by employing the *power series solutions* of differential equations.

Example 1: The geometry of a straight fin of triangular profile is described in Figure 3.8. The base temperature T_o of the fin is specified. The temperature distribution and the rate of heat transfer from this triangular fin can be determined as,

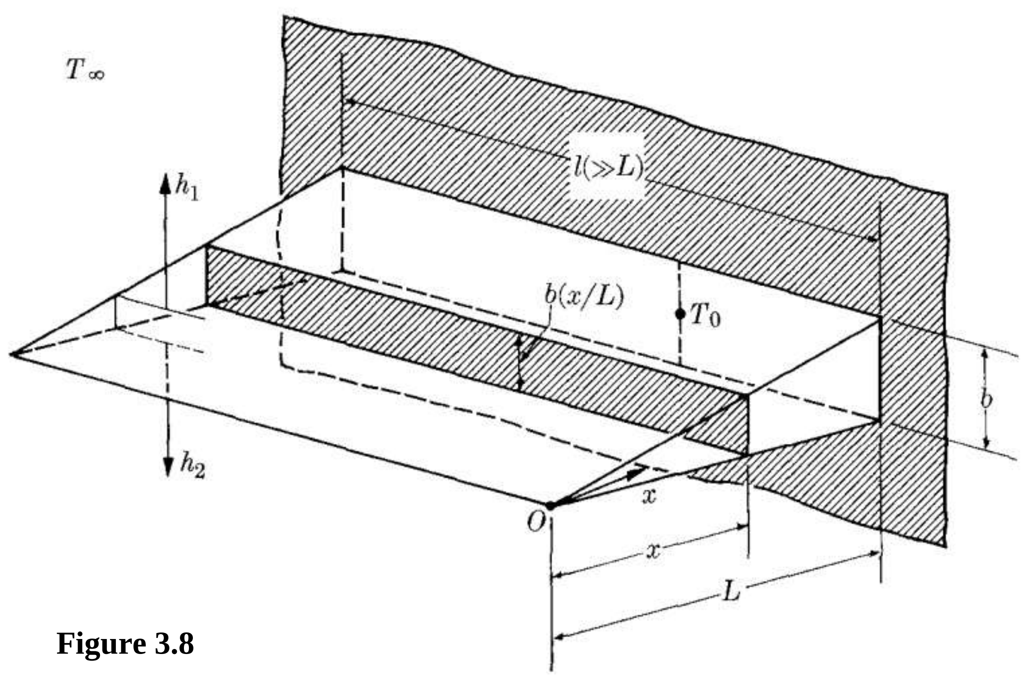


Figure 3.8

Solution:

$b/L << 1$

$L/l << 1$ (The temperature distribution of the present problem is One-dimensional or that the ends in the l-direction are insulated)

Noting from Figure 3.8 that $\theta = (T - T_\infty)$, $A_c = (b\frac{x}{L}) \times l$ and $hP = (\frac{h_1+h_2}{2}) \times 2l$, inserting these values into Eq. (3.4), $\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{h dA_s}{k\,dx}(T - T_\infty) = 0$, and rearranging the result, we get

$$\frac{d}{dx}\left(\left(b\frac{x}{L}\right) \times l\frac{d\theta}{dx}\right) - \frac{\left(\frac{h_1+h_2}{2}\right)\times 2ldx}{k\,dx}\theta = 0 \qquad \rightarrow \qquad \frac{d}{dx}\left[\left(\frac{bl}{L}x\right)\frac{d\theta}{dx}\right] - \frac{(h_1+h_2)l}{k}\theta = 0$$

$$\frac{d}{dx}\left[x\frac{d\theta}{dx}\right] - \frac{(h_1+h_2)L}{bk}\theta = 0$$

$$\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) - m^2\theta = 0 \qquad \rightarrow \qquad x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - m^2x\theta = 0 \qquad (\times X) \qquad \text{(3-21)}$$

where, $m^2 = (h_1+h_2)L/kb$. Comparison of Eq. (3.21) with the general form of *Bessel function* gives,

$$X^2\frac{d^2y}{dx^2} + [(1-2A)X - 2BX^2]\frac{dy}{dx} + [C^2D^2X^{2C} + B^2X^2 - B(1-2A)X + A^2 - C^2n^2]y = 0$$

$$1 - 2A = 1 \qquad \xrightarrow{yields} A = 0$$

$$2B = 0 \qquad \xrightarrow{yields} B = 0$$

$$2C = 1 \qquad \xrightarrow{yields} C = \frac{1}{2}$$

$$C^2D^2 = -m^2 \qquad (\tfrac{1}{2})^2D^2 = -m^2 \qquad D = \sqrt{-m^2} \times \sqrt{4} \qquad \xrightarrow{yields} D = im \times 2$$

$$C^2n^2 = 0 \qquad \xrightarrow{yields} n = 0$$

$$y = X^A e^{BX}[C_1 I_n(DX^C) + C_2 K_n(DX^C)] \qquad \text{(Because } D \text{ is } \textit{imaginary}\text{)}$$

$$\theta_{(x)} = C_1 I_0\left(2mX^{\frac{1}{2}}\right) + C_2 K_0\left(2mX^{\frac{1}{2}}\right) \qquad \text{(3-22)}$$

Boundary conditions:

$$x = 0 \quad \rightarrow \quad \theta_{(0)} = finite\ value\ of\ temperature\ at\ the\ fin\ tip$$

Since we have from Figure 3.8 (d), $lim_{x\to 0}\, K_o(x) \to \infty$

Therefore, the finiteness of tip temperature implies C_2= 0.

Next, the use of the base temperature, $x = L \quad \to \quad \theta_{(L)} = \theta_o = (T_o - T_\infty)$

So, $C_1 = \theta_o / I_o\left(2mL^{1/2}\right)$. Inserting the values of C_l and C_2 into Eq. (3.22), we find that the temperature distribution in the fin is

$$\frac{\theta_{(x)}}{\theta_o} = \frac{I_o(2mx^{1/2})}{I_o(2mL^{1/2})} \qquad (3\text{-}23)$$

Again, the heat transfer from the fin may conveniently be obtained by considering the conduction through its base. Thus

$$q = -[-kA(d\theta/dx)_{x=L}]$$

which may be evaluated from Eq. (3.23). It follows, in terms of ($\xi = 2mx^{1/2}$), that

$\frac{d}{dx}[I_o(\xi)] = \frac{d}{d\xi}[I_o(\xi)]\frac{d\xi}{dx} = \frac{m}{x^{1/2}} I_1(2mx^{1/2})$, and the heat transfer from the fin is

$$q = kA\theta_o \frac{(m/L^{1/2})\, I_1(2mL^{1/2})}{I_o(2mL^{1/2})} \quad \to \quad \frac{q}{\frac{kA\theta_o}{L}} = \frac{(mL^{1/2})\, I_1(2mL^{1/2})}{I_o(2mL^{1/2})}$$

Example 2: Consider a straight fin of parabolic profile as shown in Figure 3.9. The thermal conductivity, base thickness, and length of the fin are *k*, 2*b*, and *L*, respectively. The heat transfer coefficient is *h* and the ambient temperature T_∞. Find the steady temperature and the total heat transfer from the fin, assuming that parabola is given by $y = Cx^{1/2}$, where *C* is a constant.

Solution:

$y = Cx^{1/2}$, at $x = L$ then $y = b$

$C = \frac{b}{\sqrt{L}} \quad \to \quad y = \left(\frac{b}{\sqrt{L}}\right) x^{1/2}$

$A_c = 2y \times 1 = 2y$

$A_s = Pdx = 2\,\underbrace{(2y+1)}_{\substack{very\ small\\ negligible}}\,dx = 2dx$

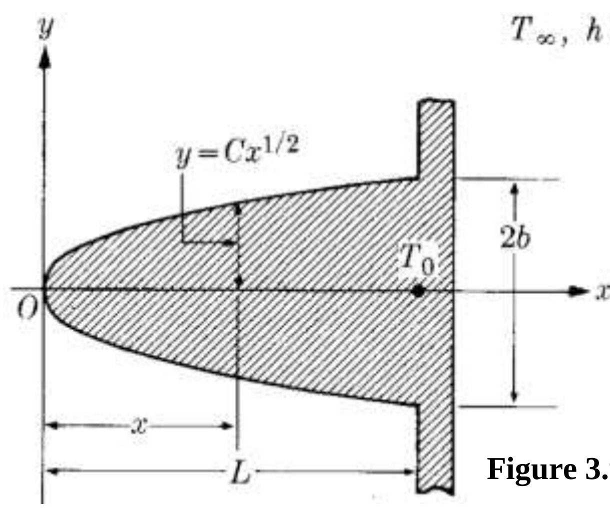


Figure 3.9

From energy balance,

$$q_x = q_{x+dx} + q_{conv.}$$

$$\frac{\partial}{\partial x}\left(-kA_c\frac{\partial T}{\partial x}\right)dx + hPdx(T-T_\infty) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{2b}{\sqrt{L}}x^{1/2}k\frac{\partial T}{\partial x}\right) - 2h(T-T_\infty) = 0 \qquad \text{Let } \theta = (T-T_\infty)$$

$$\frac{\partial}{\partial x}\left(x^{1/2}\frac{\partial\theta}{\partial x}\right) - \frac{h\sqrt{L}}{kb}\theta = 0 \qquad \text{Let } m^2 = \frac{h\sqrt{L}}{kb}$$

Or Eq. (3.4) can be applied directly as,

$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{hdA_s}{k\,dx}(T-T_\infty) = 0$$

$$\frac{d}{dx}\left(2y\frac{d\theta}{dx}\right) - \frac{h2dx}{k\,dx}\theta = 0 \qquad \frac{d}{dx}\left(2\left(\frac{b}{\sqrt{L}}\right)x^{1/2}\frac{d\theta}{dx}\right) - \frac{2h}{k}\theta = 0$$

$$\frac{d}{dx}\left(x^{1/2}\frac{d\theta}{dx}\right) - \frac{h\sqrt{L}}{kb}\theta = 0$$

Therefore,

$$x^{1/2}\frac{\partial^2\theta}{\partial x^2} + \frac{1}{2\sqrt{x}}\frac{\partial\theta}{\partial x} - m^2\theta = 0 \qquad \times(x\sqrt{x})$$

$$x^2\frac{\partial^2\theta}{\partial x^2} + \frac{1}{2}x\frac{\partial\theta}{\partial x} - m^2x^{3/2}\theta = 0$$

By comparing the above equation with *Bessel functions* as,

$$X^2\frac{d^2y}{dx^2} + [(1-2A)X - 2BX^2]\frac{dy}{dx} + [C^2D^2X^{2C} + B^2X^2 - B(1-2A)X + A^2 - C^2n^2]y = 0$$

Therefore,

$$1-2A = \frac{1}{2} \quad \xrightarrow{yields} A = \frac{1}{4}$$

$$2B = 0 \quad \xrightarrow{yields} B = 0$$

$$2C = \frac{3}{2} \quad \xrightarrow{yields} C = \frac{3}{4}$$

$$C^2D^2 = -m^2 \qquad \left(\frac{3}{4}\right)^2 D^2 = -m^2 \qquad D = \sqrt{-m^2}\times\sqrt{\left(\frac{4}{3}\right)^2} \quad \xrightarrow{yields} D = im\times\frac{4}{3}$$

$$A^2 - C^2n^2 = 0 \qquad (\frac{1}{4})^2 - (\frac{3}{4})^2n^2 = 0 \qquad \frac{1}{16}\times\frac{16}{9} = n^2 \quad \xrightarrow{yields} n = \frac{1}{3}$$

$$\theta_{(x)} = X^{\frac{1}{4}}\left[C_1 I_{\frac{1}{3}}\left(\frac{4}{3}mX^{\frac{3}{4}}\right) + C_2 K_{\frac{1}{3}}\left(\frac{4}{3}mX^{\frac{3}{4}}\right)\right]$$

Let, $u = \frac{4}{3} m X^{\frac{3}{4}}$ $X = \left(\frac{3}{4m}\right)^{\frac{4}{3}} u^{\frac{4}{3}}$ Now, substituting into $\theta_{(x)}$equation as,

$$\theta_{(u)} = \left(\left(\frac{3}{4m}\right)^{\frac{4}{3}} u^{\frac{4}{3}}\right)^{\frac{1}{4}} [C_1 I_{\frac{1}{3}}(u) + C_2 K_{\frac{1}{3}}(u)]$$

$$\theta_{(u)} = \left(\left(\frac{3}{4m}\right)^{\frac{1}{3}} u^{\frac{1}{3}}\right) [C_1 I_{\frac{1}{3}}(u) + C_2 K_{\frac{1}{3}}(u)]$$

Boundary conditions:

$x = 0 \quad \rightarrow \quad u = 0 \quad \rightarrow \quad \theta_{(0)} = finite\ value\ of\ temperature$

Since we have from Figure 3.8 (d), $lim_{u \to 0} K_{\frac{1}{3}}(u) \rightarrow \infty$

The finiteness of tip temperature implies C_2= 0.

Next, the use of the base temperature, $x = L \ \rightarrow \ u = \frac{4}{3} m L^{\frac{3}{4}} \ \rightarrow \ \theta_{(L)} = \theta_o$

$$\theta_o = \left(\left(\frac{3}{4m}\right)^{\frac{1}{3}} \left(\frac{4}{3} m L^{\frac{3}{4}}\right)^{\frac{1}{3}}\right) C_1 I_{\frac{1}{3}}\left(\frac{4}{3} m L^{\frac{3}{4}}\right) \qquad \theta_o = L^{\frac{1}{4}} C_1 I_{\frac{1}{3}}\left(\frac{4}{3} m L^{\frac{3}{4}}\right)$$

$$\frac{\theta_o}{L^{\frac{1}{4}} I_{\frac{1}{3}}\left(\frac{4}{3} m L^{\frac{3}{4}}\right)} = C_1 \qquad (3\text{-}24)$$

Inserting the values of C_l and C_2 into $\theta_{(u)}$ equation, we find that the temperature distribution in the fin is

$$\theta_{(u)} = \left(\left(\frac{3}{4m}\right)^{\frac{1}{3}} u^{\frac{1}{3}}\right) \left[\frac{\theta_o}{L^{\frac{1}{4}} I_{\frac{1}{3}}\left(\frac{4}{3} m L^{\frac{3}{4}}\right)} I_{\frac{1}{3}}(u)\right] \qquad \frac{\theta_{(u)}}{\theta_o} = \left(\left(\frac{3}{4m}\right)^{\frac{1}{3}} u^{\frac{1}{3}}\right) \left[\frac{L^4 I_{\frac{1}{3}}(u)}{I_{\frac{1}{3}}\left(\frac{4}{3} m L^{\frac{3}{4}}\right)}\right] \qquad (3\text{-}25)$$

$$\frac{\theta_{(x)}}{\theta_o} = \left(\frac{X}{L}\right)^{\frac{1}{4}} \left[\frac{I_{\frac{1}{3}}\left(\frac{4}{3} m X^{\frac{3}{4}}\right)}{I_{\frac{1}{3}}\left(\frac{4}{3} m L^{\frac{3}{4}}\right)}\right] \qquad (3\text{-}26)$$

Again, the heat transfer from the fin may conveniently be obtained by considering the conduction through its base. Thus

$$q = -[-kA(d\theta/dx)_{x=L}]$$

which may be evaluated from Eq. (3.26). It follows, in terms of ($\xi = \frac{4}{3} m x^{\frac{3}{4}}$), that

$\frac{d}{dx}\left[I_{\frac{1}{3}}(\xi)\right] = \frac{d}{d\xi}\left[I_{\frac{1}{3}}(\xi)\right]\frac{d\xi}{dx} = \frac{4m}{3x^{1/4}} I_{\frac{4}{3}}(\frac{4}{3}mx^{\frac{3}{4}})$, and the heat transfer from the fin is

$$\left.\frac{d\theta}{dx}\right|_{x=L} = \frac{\theta_o L^4}{I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)} \; [L^{\frac{1}{4}} \times \frac{4m}{3L^{1/4}} I_{\frac{4}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) + I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) \times \frac{L^{\frac{-3}{4}}}{4}]$$

$$\left.\frac{d\theta}{dx}\right|_{x=L} = \frac{\theta_o L^4}{I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)} \; [\frac{4m}{3} I_{\frac{4}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) + \frac{1}{4L^{\frac{3}{4}}} I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)]$$

$$q = kA \; \frac{\theta_o L^4}{I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)} \; [\frac{4m}{3} I_{\frac{4}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right) + \frac{1}{4L^{\frac{3}{4}}} I_{\frac{1}{3}}\left(\frac{4}{3}mL^{\frac{3}{4}}\right)] \qquad (3\text{-}27)$$

3.2.4 Extended Surface Efficiency

The maximum rate at which a fin could dissipate energy is the rate that would exist if the entire fin surface were at the base temperature. However, since any fin is characterized by a finite conduction resistance, a temperature gradient must exist along the fin and the preceding condition is an idealization. A logical definition of *extended surface efficiency* is therefore, the ratio of the actual to a hypothetical (Ideal) heat transfer:

$$\eta_f = \frac{\text{Actual heat transfer from extended surface}}{\text{Ideal Heat transfer from extended surface}} = \frac{q_{fin}}{q_{Max.}} \qquad (3\text{-}28)$$

Ideal heat transfer from extended surface, it is the heat transfer at the base temperature, $q_{Ideal} = hA_s(T_b - T_\infty)$ where A_s total surface area of the fin

The denominator of Eq. (3.28) denotes the heat transfer from an area of the wall equivalent to the base area of the extended surface; the heat transfer to be evaluated by numerator and denominator together is based on the same temperature difference, base minus ambient. Since the temperature of a wall and the heat transfer coefficient between the wall and the ambient are somewhat changed when an extended surface is attached to the wall, the efficiency defined by Eq. (3.28) is quite approximate. The error involved in this approximation depends on the length of the extended surfaces and the space between them. Therefore, rather than to demonstrate the increased heat transfer from a wall by the use of extended surfaces, this efficiency may better be employed to compare different extended surfaces. The particular values of these efficiencies for specific cases are,

Case D: $$\eta_f\Big|_{\text{Infinite fin (L}\rightarrow\infty)} = \frac{\theta_o\sqrt{hPkA_c}}{\theta_o hA_c} = \sqrt{\frac{kP}{hA_c}} \qquad (3\text{-}29)$$

Case B: $$\eta_f\Big|_{\text{Adiabatic fin tial}} = \frac{\theta_o\sqrt{hPkA_c}\tanh mL}{\theta_o hA_c} = \sqrt{\frac{kP}{hA_c}}\tanh mL \qquad (3\text{-}30)$$

The efficiencies of extended surfaces have been extensively investigated in the literature. In practice, however, the technology involved may be a more important consideration than finding a 5-10% more efficient profile which is expensive to manufacture.

In contrast to the fin efficiency η_f, which characterizes the performance of a single fin, the overall surface efficiency η_o characterizes an array of fins and the base surface to which they are attached. Representative arrays are shown in Figure 3.10, where S designates the fin pitch. In each case the overall efficiency is defined as,

$$\eta_f = \frac{q_t}{q_{max}} = \frac{q_t}{hA_f\theta_b} \qquad (3\text{-}31)$$

where q_t is the total heat rate from the surface area A_t associated with both the fins and the exposed portion of the base (often termed the prime surface). If there are N fins in the array, each of surface area A_f, and the area of the prime surface is designated as A_b, the total surface area is

$$A_t = NA_f + A_b \qquad (3\text{-}32)$$

The maximum possible heat rate would result if the entire fin surface, as well as the exposed base, were maintained at T_b. The total rate of heat transfer by convection from the fins and the prime (unfinned) surface may be expressed as

$$q_t = N\,\eta_f hA_f\theta_b + hA_b\theta_b \qquad (3\text{-}33)$$

where the convection coefficient h is assumed to be equivalent for the finned and prime surfaces and η_f is the efficiency of a single fin. Hence

$$q_t = h[N\,\eta_f A_f + (A_t - NA_f)]\theta_b = hA_t[1 - \frac{NA_f}{A_t}(1 - \eta_f)]\theta_b \qquad (3\text{-}34)$$

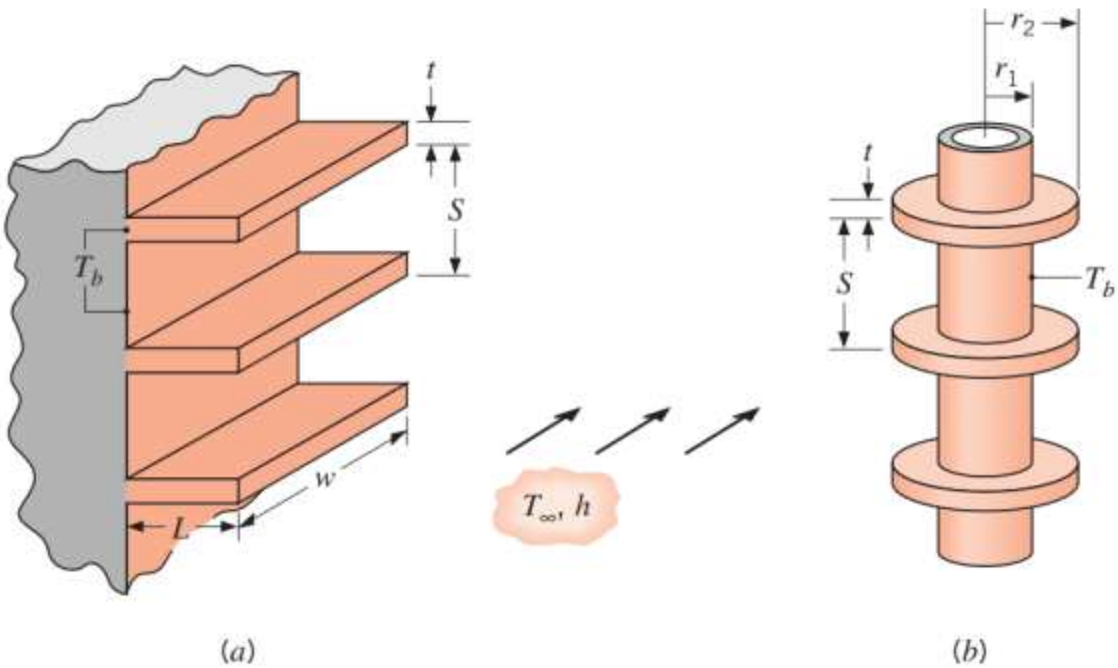


Figure 3.10: Representative fin arrays. (a) Rectangular fins. (b) Annular fins.

Substituting Equation (3.34) into (3.31), it follows that

$$\eta_o = 1 - \frac{NA_f}{A_t}(1 - \eta_f) \qquad (3\text{-}35)$$

From knowledge of η_o, Equation 3.31 may be used to calculate the total heat rate for a fin array.

3.2.5 Extended Surface Effectiveness

Recall that fins are used to increase the heat transfer from a surface by increasing the effective surface area. However, the fin itself represents a conduction resistance to heat transfer from the original surface. For this reason, there is no assurance that the heat transfer rate will be increased through the use of fins. An assessment of this matter may be made by evaluating the fin effectiveness ε_f. It is defined as *the* *ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin*. Therefore

$$\varepsilon_f = \frac{\text{Actual heat transfer from extended surface}}{Rate\ of\ heat\ transfer\ from\ the\ base\ area\ (without\ fin)} = \frac{q_f}{hA_{c,b}\theta_b} \qquad (3\text{-}36)$$

where $A_{c,b}$ is the fin cross-sectional area at the base. In any rational design the value of ε_f should be as large as possible, and in general, the use of fins may rarely be justified unless $\varepsilon_f \geq 2$.

Subject to any one of the four tip conditions that have been considered, the effectiveness for a fin of uniform cross section may be obtained by dividing the appropriate expression for q_f by $hA_{c,b}\theta_b$. Although the installation of fins will alter the surface convection coefficient, this effect is commonly neglected. Hence, assuming the convection coefficient of the finned surface to be equivalent to that of the unfinned base, it follows that, for the infinite fin approximation (**Case D**), the result is,

$$\varepsilon_f = \sqrt{\frac{kP}{hA_c}} \qquad (3\text{-}37)$$

Several important trends may be inferred from this result. Obviously, fin effectiveness is enhanced by the choice of a material of high thermal conductivity. Aluminum alloys and copper come to mind. However, although copper is superior from the standpoint of thermal conductivity, aluminum alloys are the more common choice because of additional benefits related to lower cost and weight. Fin effectiveness is also enhanced by increasing the ratio of the perimeter to the cross-sectional area. For this reason, the use of thin, but closely spaced fins is preferred, with the proviso that the fin gap not be reduced to a value for which flow between the fins is severely impeded, thereby reducing the convection coefficient. Equation 3.37 also suggests that the use of fins can be better justified under conditions for which the convection coefficient *h* is small. Hence it is evident that the need for fins is stronger when the fluid is a gas rather than a liquid and when the surface heat transfer is by free convection. If fins are to be used on a surface separating a gas and a liquid, they are generally placed on the gas side, which is the side of lower convection coefficient. A common example is the tubing in an automobile radiator. Fins are applied to the outer tube surface, over which there is flow of ambient air (*small h*), and not to the inner surface, through which there is flow of water (*large h*). Note that, if $\varepsilon_f \geq 2$ is used as a criterion to justify the implementation of fins, Equation 3.37 yields the requirement that $(kP/hA_c) \geq 4$.

We can draw several important conclusions from equation (3.37) for consideration in the design and selection of the fins:

1. The thermal conductivity of the fin material (*k*) should be as high as possible. The most widely used fins are made of aluminum.
2. The ratio of the perimeter to the cross-section area of the fin should be as high as possible. This condition is satisfied by thin plate fins or slender pin fins.
3. The use of fins is most effective in applications involving a low convective heat transfer coefficient (*h*). In liquid-to-gas heat exchanger such as the car radiator, fins are placed on the gas side.

Homework 2: Consider a straight fin of parabolic profile as shown in Figure 3.11. The thermal conductivity, base thickness, and length of the fin are k, $2b$, and L, respectively. The heat transfer coefficient is h and the ambient temperature T_∞. Find the steady temperature of and the total heat transfer from the fin, assuming that parabola is given by $y = Cx^2$, where C is a constant.

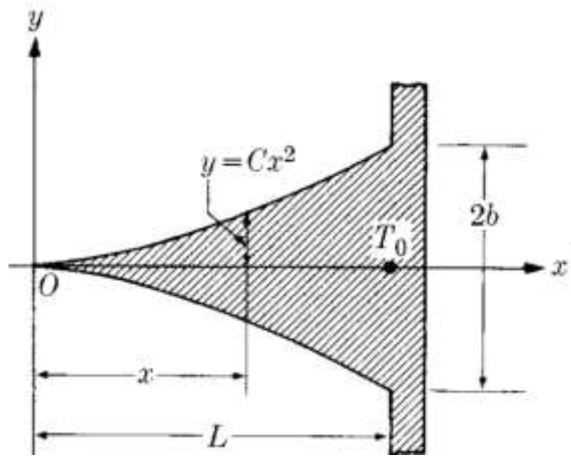


Figure 3.11

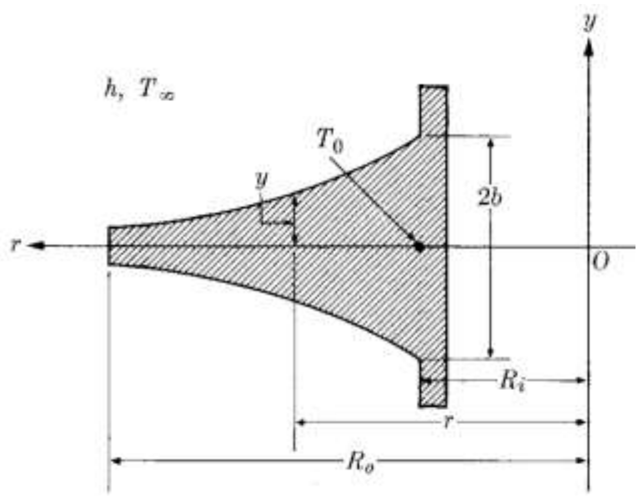


Figure 3.12

Homework 3: Consider a straight fin of parabolic profile as shown in Figure 3.12. The thermal conductivity, base thickness, and inner and outer radii of the fin are k, $2b$, R_i and R_o, respectively. The heat transfer coefficient is h and the ambient temperature T_∞. Find the steady temperature of and the total heat transfer from the fin, assuming that hyperbola is given by (a) $yr^{1/2} = C$, (b) $yr^2 = C$ where C is a constant.

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Advanced Heat Transfer/ I Conduction and Radiation

Handout Lectures for MSc. / Power Chapter Four/ Steady-State Two-Dimensional Conduction Heat Transfer

Course Tutor

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Chapter Four

Steady-State Two-Dimensional Conduction Heat Transfer

4.1 Introduction

In many cases such problems are grossly oversimplified if a one-dimensional treatment is used, and it is necessary to account for multidimensional effects. In this chapter, we will focus on "*analytical method*" for treating two-dimensional systems under steady-state conditions.

4.2 Boundary-value problems and characteristic-value problems

Consider an ordinary differential equation of second order which may result from the differential formulation of a steady one-dimensional conduction problem. The solution of this equation involves two arbitrary constants which are determined by two conditions, each specified at one boundary of the problem. Problems of this type are called *boundary-value problems* to distinguish them from *initial-value problems*, in which all conditions are specified at one location. Reconsider the differential equation

$$\frac{d^2y}{dx^2} + y = 0 \tag{4-1}$$

Assume that this homogeneous equation involves a parameter "λ" as

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0 \tag{4-2}$$

And is subject to homogeneous boundary conditions

$y(0) = 0$, and $y(L) = 0$ then the general solution of Equation (4-2) is

$$y = C_1 \sin \lambda x + C_2 \cos \lambda x \tag{4-3}$$

The use of ($y(0) = 0$) results in $C_2 = 0$ and , $y = C_1 \sin \lambda x$

From $(y(L) = 0)$, combined with $y = C_1 \sin \lambda x$, gives $C_1 \sin \lambda L = 0$. The problem has nontrivial solutions only if λ satisfies the $(\sin \lambda L = 0)$. Therefore,

$$\lambda_n = \frac{n\pi}{L}, \quad \text{where } n = 1, 2, 3, \ldots\ldots, \tag{4-4}$$

And the corresponding solutions of $(y = C_1 \sin \lambda x)$ are,

$$y = C_1 \varphi_n(x)\,, \quad \varphi_n(x) = \sin(\frac{n\pi}{L})\,x \tag{4-5}$$

Note that no new solutions are obtained when "*n*" assumes negative integer values. Thus, the foregoing *boundary-value problem* has no solution other than the trivial solution *y*= 0, unless λ assumes one of the characteristic values given by Equation 4-4. Corresponding to each characteristic value of λ_n there exists a characteristic function $\varphi_n(x)$ given by Equation 4-5, such that any constant multiple of this function is a solution of the problem. It is important to note that the *boundary-value problem* given by

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0\,; \qquad y(0) = 0\,, \quad y(L) = 0$$

has no solution other than the trivial solution *y*= 0 corresponding to λ= 0. Hence there does not exist any set of characteristic values and characteristic functions for this problem. This illustrates the fact that a *boundary-value problem* may or may not be a *characteristic-value problem*. A *boundary-value problem* is a *characteristic-value problem* when it has particular solutions that are periodic in nature; the period and amplitude of these solutions may or may not be constant.

Therefore, in the next three sections the general properties of characteristic functions are investigated.

4.3 Orthogonality of Characteristic Functions

By definition, two functions $\varphi_n(x)$ and $\varphi_m(x)$ are said to be orthogonal with respect to a weighting function $w(x)$, over a finite interval (*a*, *b*), if the integral of the product $w(x)\varphi_n(x)\varphi_m(x)$ over that interval vanishes as

$$\int_a^b w(x)\varphi_n(x)\varphi_m(x)dx = 0\,, \qquad \text{where } m \neq n \tag{4-6}$$

Furthermore, a set of functions is said to be *orthogonal* in (*a*, *b*) if all pairs of distinct functions in the set are *orthogonal* in (*a*, *b*). The word *orthogonality* comes from vector analysis. Let $\varphi_m(x_i)$ denote a vector in 3D space whose rectangular components are $\varphi_m(x_1)$, $\varphi_m(x_2)$, and $\varphi_m(x_3)$. Two vectors, $\varphi_m(x_i)$ and $\varphi_n(x_i)$, are said to be *orthogonal*, or perpendicular to each other, if

$$\varphi_m(x_i)\,.\,\varphi_n(x_i) = \sum_{i=1}^{3} \varphi_m(x_i)\,.\,\varphi_n(x_i) = 0 \tag{4-7}$$

When the units of length on the coordinate axes vary from one axis to another, the foregoing scalar product assumes the form

$$\varphi_m(x_i)\,.\,\varphi_n(x_i) = \sum_{i=1}^{N} w(x_i)\,\varphi_m(x_i)\,\varphi_n(x_i) = 0 \tag{4-8}$$

Where the weighting numbers $w(x_1)$, $w(x_2)$, and $w(x_3)$ depend upon the units of length used along the three axes. The vectors in an N-Dimensional space having components $\varphi_m(x_i)$, $\varphi_n(x_i)$, *i*= 1, 2, 3, ……, *N* are said to be *orthogonal* with respect to the weighting numbers $w(x_i)$.

It will now be shown that the *characteristic functions* of a *characteristic-value problem* are *orthogonal* over a finite interval with respect to a weighting function. To establish this fact, consider the *characteristic-value problem* composed of the linear homogenous second-order differential equation of the general form

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + [f_2(x) + \lambda^2 f_3(x)]y = 0 \tag{4-9}$$

This equation, multiplied through by the factor ($e^{[\int f_1(x)dx]} = p(x)$) and with the functions defined as $f_2(x)p(x) = q(x)$ and $f_3(x)p(x) = w(x)$, may be rearranged in the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda^2 w(x)]y = 0 \tag{4-10}$$

Which is more convenient for the following discussion.

Let λ_m, λ_n be any two distinct characteristic numbers, that is, $m \neq n$ and let $\varphi_m(x)$, $\varphi_n(x)$ be the corresponding characteristic functions. Since $y = \varphi_m(x)$ and $y = \varphi_n(x)$ are solutions of Equation (4-10),

$$\frac{d}{dx}\left(p\,\frac{d\varphi_m}{dx}\right) + (q + \lambda_m^2 w)\varphi_m = 0,$$

$$\frac{d}{dx}\left(p\,\frac{d\varphi_n}{dx}\right) + (q + \lambda_n^2 w)\varphi_n = 0.$$

Multiplying the first equation by φ_n and the second by φ_m, then subtracting the second of the resulting equations from the first one gives

$$\varphi_n \frac{d}{dx}\left(p\,\frac{d\varphi_m}{dx}\right) - \varphi_m \frac{d}{dx}\left(p\,\frac{d\varphi_n}{dx}\right) + (\lambda_m^2 - \lambda_n^2) w \varphi_m \varphi_n = 0.$$

Integrating this equation over the finite interval (a, b) yields

$$(\lambda_n^2 - \lambda_m^2)\int_a^b w\varphi_m\varphi_n\,dx = \int_a^b \left[\varphi_n \frac{d}{dx}\left(p\,\frac{d\varphi_m}{dx}\right) - \varphi_m \frac{d}{dx}\left(p\,\frac{d\varphi_n}{dx}\right)\right] dx,$$

and integration by parts for the right-hand member results in

$$(\lambda_n^2 - \lambda_m^2)\int_a^b w\varphi_m\varphi_n\,dx = \left\{p(x)\left[\varphi_n(x)\,\frac{d\varphi_m(x)}{dx} - \varphi_m(x)\,\frac{d\varphi_n(x)}{dx}\right]\right\}\Bigg|_a^b \qquad (4\text{-}11)$$

Since both $y = \varphi_m(x)$ and $y = \varphi_n(x)$ are particular solutions of Equation 4-10, the right-hand side of Equation 4-11 vanishes when one of the following conditions is prescribed at each end of the interval (a, b):

$$y = 0 \qquad (4\text{-}12)$$

$$\frac{dy}{dx} = 0 \qquad (4\text{-}13)$$

$$\frac{dy}{dx} + By = 0 \qquad (4\text{-}14)$$

Where B is an arbitrary parameter.

The fact that Equation (4-11) vanishes when Equation (4-14) is satisfied may be clarified by rearranging the right-hand member of Equation (4-11) in the form

$$\varphi_n \acute{\varphi}_m - \varphi_m \acute{\varphi}_n = \varphi_n \acute{\varphi}_m - \varphi_m \acute{\varphi}_n \pm B\varphi_m \varphi_n = \varphi_n(\acute{\varphi}_m + B\varphi_m) - \varphi_m(\acute{\varphi}_n + B\varphi_n) \quad (4\text{-}15)$$

Particularly, if $p(x) = 0$ when $x = a$ or $x = b$, the right-hand side of Equation (4-11) vanishes, and the condition given by Equation 4-12, 4-13, or 4-14 satisfied at $x = a$ or $x = b$ can be dropped from the problem provided y and $(\frac{dy}{dx})$ are finite at that point. If $p(b) = p(a)$, the *orthogonality* continues to exist when the boundary conditions are replaced by the conditions $y(b) = y(a)$ and $\acute{y}(b) = \acute{y}(a)$, which are called the periodic boundary conditions.

As an example, reconsider the *characteristic-value problem* given by Equation (4-1). Comparison of Equations (4-1) and (4-10) gives $w(x) = 1$, and the condition of the *orthogonality* for this problem is

$$\int_0^L \varphi_m(x)\, \varphi_n(x) dx = \int_0^L \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L})\, dx = 0 \qquad m \neq n \quad (4\text{-}16)$$

Which can also be verified independently by direct integration.

We wish to expand an arbitrary function *f*(*x*) into a series of this set as

$$f(x) = b_o\varphi_o(x) + b_1\varphi_1(x) + b_2\varphi_2(x) + \cdots = \sum_{n=0}^{\infty} b_n \varphi_n(x) \quad (4\text{-}17)$$

By multiplying both sides of Equations (4-17) by $w(x)\varphi_m(x)$ and integrating the result over the interval with the assumption that the integral of the infinite sum is equivalent to the sum of the integrals,

$$\int_a^b w(x) f(x)\, \varphi_m(x) dx = \sum_{n=0}^{\infty} A_n \int_a^b w(x)\varphi_n(x)\varphi_m(x) dx \quad (4\text{-}18)$$

All terms in the sum on the right of Equation 4-18 are *zero* except the term corresponding to *n* = *m*.

$$A_n = \frac{\int_a^b w(x) f(x) \varphi_n(x) dx}{\int_a^b w(x) \varphi^2{}_n(x) dx} \quad (4\text{-}19)$$

4.4 Fourier series

In general, one must somewhere in the problem, express a function (for example *f*(*x*)) by a series of eigen functions (for example $\sin(n\pi x)$). More generally, if the eigen functions are denoted by $\varphi_n(x)$ the expression is then given by;

$$f(x) = \sum_{n=1}^{\infty} A_n \, \varphi_n(x) \qquad (4\text{-}20)$$

$\varphi_n(x)$ can usually be expected to be orthogonal with respect to some weighting function *w*(*x*). In other words;

$$\int_0^1 w(x)\, \varphi_n(x) dx = \begin{cases} 0 & (for\ m \neq n) \\ C & (for\ m = n) \end{cases} \qquad (4\text{-}21)$$

Multiply *f*(*x*) by $(w(x)\ \varphi_m(x))$ and integrate;

$$\int_0^1 w(x) f(x)\, \varphi_m(x) dx = \int_0^1 \sum_{n=1}^{\infty} A_n\, w(x)\, \varphi_n(x)\varphi_m(x) dx =$$

$$\sum_{n=1}^{\infty} A_n \int_0^1 w(x)\varphi_n(x)\varphi_m(x) dx$$ (Integral = 0 for $m \neq n$, and = C for $m = n$)

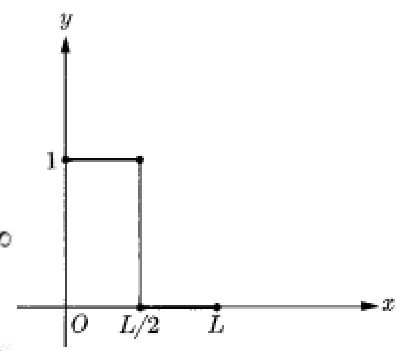
Thus, $\int_0^1 w(x) f(x)\, \varphi_m(x) dx = A_m C$

$$\frac{1}{C}\int_0^1 w(x) f(x)\, \varphi_m(x) dx = A_m \qquad (4\text{-}22)$$

Thus the coefficients have been found since all functions in the integral are known and the integral can be evaluated.

Example 1:

Consider the Fourier *sine* series of the function as;



$$f(x) = \begin{cases} 0, & -\infty < x < 0 \quad \text{and} \quad L/2 < x < \infty \\ 1, & 0 < x < L/2 \end{cases}$$

over the interval $(0, L)$ (Fig. 4–2). The coefficients of the series are

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}\right) x\, dx = \frac{2}{L}\int_0^{L/2} 1 \cdot \sin\left(\frac{n\pi}{L}\right) x\, dx$$

$$= \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right);$$

hence the series is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{L}\right) x.$$

Example 2: Consider now the Fourier cosine series of the previous example, where the coefficients of the series are

Multiply each side by $cos\ (m\pi x)$:

$\int_o^1 x\ cosm\pi xdx = \int_o^1 \sum_{n=o}^{\infty} B_n Cosn\pi x\ Cosm\pi xdx = \sum_{n=o}^{\infty} B_n \int_0^1 Cosn\pi xcosm\pi xdx$

$\int_o^1 xcosm\pi xdx = B_m \int_o^1 cos^2 m\pi xdx \qquad (\ cos^2 = \frac{1}{2}(1+\cos)\ )$

For $m = o \longrightarrow \int_o^1 xdx = B_0 \int_0^1 dx = B_0 \quad B_0 = \frac{x^2}{2}]_0^1 = \frac{1}{2}$

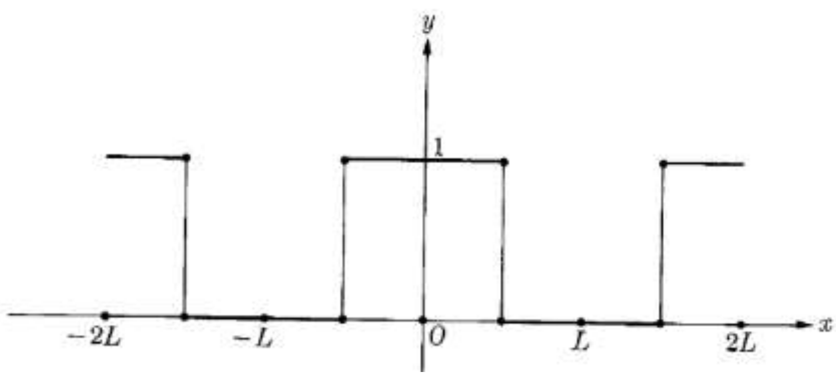
For $m \neq o$; $\int_o^1 xcosm\pi xdx = \frac{B_m}{m\pi}\int_o^1 cos^2 m\pi x\ \ d(m\pi x) = \frac{B_m}{m\pi}(\frac{m\pi x}{2} + \frac{1}{4}$ $\sin 2m\pi x)]_0^1 = \frac{B_m}{2}$

For $m \neq 0 \quad B_m = 2\int_o^x xcosm\pi xdx = \frac{2}{m^2\pi^2}(\cos(m\pi) - 1)$

$g(x) = x = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\ (\cos m\pi - 1)cosn\pi x$

Example 3:

Express a function *f*(*x*), which is piecewise continuous in the interval (-*L*, *L*), in terms of both *sine* and *cosine* having the common period 2*L* (where the function repeats its behavior periodically for all values of *x* as shown in Figure below).



So far, we have seen that any piecewise continuous function can be expressed in the interval $(0, L)$ by a series consisting of sines or cosines with the common period $2L$. When the function is odd, the sine series representation is valid in the interval $(-L, L)$, whereas for an even function the cosine series representation holds in the same interval.

Noting that

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)],$$

where the function in the first brackets is even and that in the second is odd, we arrive at the fact that an arbitrary function can be expressed as the sum of an even function and an odd function. Hence

$$f(x) = f_e(x) + f_o(x).$$

Expressing $f_e(x)$ in terms of cosines and $f_o(x)$ in terms of sines in the interval $(-L, L)$, we have

$$f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos (n\pi/L)x,$$

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin (n\pi/L)x,$$

where

$$a_0 = \frac{1}{L}\int_0^L f_e(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f_e(x)\cos\left(\frac{n\pi}{L}\right)x\,dx,$$

$$b_n = \frac{2}{L}\int_0^L f_o(x)\sin\left(\frac{n\pi}{L}\right)x\,dx.$$

Since the integrands of these equations are even functions of x, replacing $\int_0^L$ by $\frac{1}{2}\int_{-L}^{L}$ gives

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx.$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos (n\pi/L)x + b_n \sin (n\pi/L)x], \qquad -L < x < L.$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_0^{L/2} 1\cdot dx = \tfrac{1}{4},$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{L}\int_{0}^{L/2} 1\cdot\cos\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{n\pi}\sin\frac{n\pi}{2},$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{1}{L}\int_{0}^{L/2} 1\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right).$$

Therefore,

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{L}\right)x + \left(1-\cos\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{L}\right)x\right],$$

4.5 Homogeneous Problems

A differential equation is linear if it contains no products of the dependent variable or its derivatives (u^4 or uu_x terms are not permitted).

$(uu_x + vu_y = vu_{yy})$ Nonlinear Equation

A boundary condition is linear if it contains no products of the dependent variable or its derivatives.

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = C\sigma(T^4 - T_\infty^4) \quad \text{Nonlinear B.C.}$$

A differential equation is homogeneous if when it is satisfied by (u) it is also satisfied by (cu) where(c) is an arbitrary constant (c= o is special case).

$u_{xx} + q''' = u_\theta$ Non Homgeneous Equuation

$cu_{xx} + q''' = cu_\theta$

$or\ \left(u_{xx} + \frac{q'''}{c} = u_\theta\right)$ which is not identical to the original equation.

A boundary condition (different form the initial condition) is homogenous if when satisfied by (u) it is also satisfied by (cu) where (c) is an arbitrary constant;

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h(T - T_\infty) \quad \text{Non Homogenous B.C.}$$

$$-kc\frac{\partial T}{\partial x}\bigg|_{x=L} = h(cT - T_\infty)$$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h(T - \frac{T_\infty}{C}) \quad \text{which is not identical to the original B.C.}$$

A homogeneous linear problem will be defined as one in which both the differential equation and its B.Cs. are homogeneous as well as linear. Both these restrictions are essential for separation of variables to be directly applicable

A linear diff. equ. and the B.Cs. are homogeneous when all terms include either the unknown function or one of its derivatives.

4.6 The Method of Separation of Variables: Steady, Two-dimensional Cartesian Geometry

For two-dimensional, steady-state conditions with no generation and constant thermal conductivity, this form is, from Equation 1.17,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (4\text{-}1)$$

To appreciate how the method of separation of variables may be used to solve two-dimensional conduction problems, we consider the system of Figure 4.1. Three sides of a thin rectangular plate or a long rectangular rod are maintained at a constant temperature T_1, while the fourth side is maintained at a constant temperature $T_2 \neq T_1$. Assuming negligible heat transfer from the surfaces of the plate or the ends of the rod, temperature gradients normal to the x–y plane may be neglected ($\partial^2 T/\partial z^2 = 0$) and conduction heat transfer is primarily in the *x*- and *y*-directions.

We are interested in the temperature distribution $T(x, y)$, but to simplify the solution we introduce the transformation,

$$\theta = \frac{T - T_1}{T_2 - T_1} \qquad (4\text{-}2)$$

By Substituting Equation 4.2 into Equation 4.1, the transformed differential equation is then,

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \qquad (4\text{-}3)$$

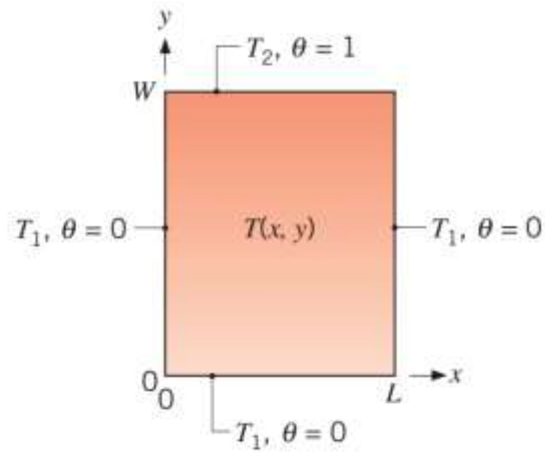


Figure 4.1: Two-dimensional conduction problems, sides of a thin rectangular plate or a long rectangular

Since the equation is second order in both *x* and *y*, two boundary conditions are needed for each of the coordinates. They are,

$\theta(0, y) = 0$, $\theta(x, 0) = 0$, $\theta(L, y) = 0$, $\theta(x, W) = 1$

Note that, through the transformation of Equation 4.2, three of the four boundary conditions are now homogeneous and the value of θ is restricted to the range from 0 to 1. We now apply the separation of variables technique by assuming that the desired solution can be expressed as the product of two functions, one of which depends only on *x* while the other depends only on *y*. The essential features of the method will now be illustrated by means of a steady two- dimensional example. Consider the *second-order* partial differential equation,

$$a_1(x)\frac{\partial^2 \theta}{\partial x^2} + a_2(x)\frac{\partial \theta}{\partial x} + a_3(x)\theta + b_1(y)\frac{\partial^2 \theta}{\partial y^2} + b_2(y)\frac{\partial \theta}{\partial y} + b_3(y)\theta = 0 \qquad (4\text{-}4)$$

A more generalized form of this equation which involves coefficients as functions of both independent variables is not suitable for the separation of variables. That is, assume the existence of a product solution of the form

$$\theta_{(x,y)} = X_{(x)}\, Y_{(y)} \qquad (4\text{-}5)$$

Where X is a function of x alone and Y is a function y. This assumption becomes meaningful when the two functions X and Y actually satisfy separate differential equations.

Introducing Eq. (4.5) into Eq. (4.4) and dividing the result by XY yields

$$\left[a_1(x)\frac{\partial^2 X}{\partial x^2} + a_2(x)\frac{\partial X}{\partial x} + a_3(x)X\right]\frac{1}{X} = -\left[b_1(y)\frac{\partial^2 Y}{\partial y^2} + b_2(y)\frac{\partial Y}{\partial y} + b_3(y)Y\right]\frac{1}{Y} = 0 \qquad (4\text{-}6)$$

It is evident that the differential equation is, in fact, separable. That is, the *left-hand* side of the equation depends only on x and the *right-hand* side depends only on y. Hence the equality can apply in general (for any x or y) only if both sides are equal to the same constant. Identifying this, as yet unknown, *separation constant* or *separation parameter* as ($+\lambda^2$) or ($-\lambda^2$), we then have

$$a_1(x)\frac{\partial^2 X}{\partial x^2} + a_2(x)\frac{\partial X}{\partial x} + [a_3(x) \pm \lambda^2]X = 0 \qquad (4\text{-}7)$$

$$b_1(y)\frac{\partial^2 Y}{\partial y^2} + b_2(y)\frac{\partial Y}{\partial y} + [b_3(y) \pm \lambda^2]Y = 0 \qquad (4\text{-}8)$$

The method of separation of variables is applicable to steady two-dimensional problems if and when,

i. One of the directions of the problem is expressed by a homogeneous differential equation subject to homogeneous boundary conditions (the homogeneous direction), while the other direction is expressed by a homogeneous differential equation subject to one homogeneous and one nonhomogeneous boundary condition (the nonhomogeneous direction).

ii. The sign of λ^2 is chosen such that the boundary-value problem of the homogeneous direction leads to a characteristic-value problem.

The solutions obtained by the separation of variables are in the form of a sum or integral, depending on whether the homogeneous direction is finite or extends to infinity, respectively.

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0 \quad (4\text{-}9)$$

$$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0 \quad (4\text{-}10)$$

The partial differential equation has been reduced to two ordinary differential equations. Note that the designation of λ^2 as a positive constant was not arbitrary. If a negative value were selected or a value of $\lambda^2 = 0$ was chosen, it would be impossible to obtain a solution that satisfies the prescribed boundary conditions. The general solutions to Equations 4.9 and 4.10 are, respectively,

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x \quad (4\text{-}11)$$

$$Y = C_3 e^{-\lambda y} + C_4 e^{+\lambda y} \quad (4\text{-}12)$$

in which case the general form of the two-dimensional solution is

$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)\left(C_3 e^{-\lambda y} + C_4 e^{+\lambda y}\right) \quad (4\text{-}13)$$

The classical method of separation of variables is restricted to linear homogeneous P.D.E.

Example 1: A two-dimensional rectangular plate is subjected to the boundary conditions shown in Figure 4.1. Derive an expression for the steady-state temperature distribution $\theta(x, y)$.

Solution:

The transformed differential equation (applying Eq. 4.3) as,

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

The boundary conditions are needed for each of the coordinates as,

$\theta(0, y) = 0\ ,\quad \theta(x, 0) = 0,\quad \theta(L, y) = 0\ ,\quad \theta(x, W) = 1$

Assume the existence of a product solution of the form (Eq. 4.5)

$\theta_{(x,y)} = X_{(x)}\, Y_{(y)}$

Where X is a function of x alone and Y is a function y. The solutions obtained by the separation of variables are in the form of a sum or integral, depending on whether the homogeneous direction is finite or extends to infinity, respectively.

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0$$

The general form of the two-dimensional solution is,

$$\theta_{(x,y)} = \left(C_1 \cos\lambda x + C_2 \sin\lambda x\right)\left(C_3 e^{-\lambda y} + C_4 e^{+\lambda y}\right) \qquad (4\text{-}14)$$

Applying the condition that $\theta(0, y)= 0$, it is evident that $C_1= 0$. In addition from the requirement that $\theta(x, 0)= 0$, we obtain

$$C_2 \sin\lambda x\,(C_3 + C_4) = 0$$

which may only be satisfied if $C_3= -C_4$. Although the requirement could also be satisfied by having $C_2= 0$, this would result in $\theta(x, y)= 0$, which does not satisfy the boundary condition $\theta(x, W)= 1$. If we now invoke the requirement that $\theta(L, y)= 0$, we obtain

$$C_2 C_4 \sin\lambda L\left(e^{\lambda y} - e^{-\lambda y}\right) = 0\ ,$$

The only way in which this condition may be satisfied (and still have a *nonzero* solution) is by requiring that assume discrete values for which $\sin\lambda L= 0$. These values must then be of the form,

$$\lambda = \frac{n\pi}{L} \qquad n = 1, 2, 3, \ldots.$$

where the integer $n= 0$ is precluded, since it implies $\theta(x,y)= 0$. The desired solution may now be expressed as

$$\theta = C_2 C_4 \sin\frac{n\pi x}{L}\left(e^{\frac{n\pi y}{L}} - e^{\frac{-n\pi y}{L}}\right)$$

Combining constants and acknowledging that the new constant may depend on *n*, we obtain

$$\theta_{(x,y)} = C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$$

where we have also used the fact that $\left(e^{\frac{n\pi y}{L}} - e^{\frac{-n\pi y}{L}}\right) = 2\sinh\left(\frac{n\pi y}{L}\right)$. In this form we have really obtained an infinite number of solutions that satisfy the differential equation and boundary conditions. However, since the problem is linear, a more general solution may be obtained from a superposition of the form

$$\theta_{(x,y)} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$$

To determine C_n we now apply the remaining boundary condition, which is of the form

$$\theta_{(x,W)} = 1 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi W}{L} \qquad (4\text{-}15)$$

Although the above equation would seem to be an extremely complicated relation for evaluating C_n, a standard method is available. It involves writing an infinite series expansion in terms of *orthogonal functions*. An infinite set of functions $g_1(x)$, $g_2(x)$, … , $g_n(x)$, … is said to be orthogonal in the domain $a \le x \le b$ if

$$\int_a^b g_m(x) g_n(x) dx = 0 \qquad m \neq n$$

Many functions exhibit *orthogonality*, including the *trigonometric functions* $sin(n\pi x/L)$ and $cos(n\pi x/L)$ for $0 \le x \le L$. Their utility in the present problem rests with the fact that any function $f(x)$ may be expressed in terms of an infinite series of orthogonal functions

$$f(x) = \sum_{n=1}^{\infty} A_n g_n(x) \qquad (4\text{-}16)$$

The form of the coefficients A_n in this series may be determined by multiplying each side of the equation by $g_m(x)$ and integrating between the limits *a* and *b*.

$\int_a^b f(x) g_m(x) dx = \int_a^b g_m(x) \sum_{n=1}^{\infty} A_n g_n(x)\, dx$

However, from above equation it is evident that all but one of the terms on the *right-hand* side of equation 4.16 must be zero, leaving us with

$$\int_a^b f(x) g_m(x) dx = A_m \int_a^b g_m^2\,(x) dx \qquad (4\text{-}17)$$

Hence, solving for A_m, and recognizing that this holds for any A_n by switching m to n:

$$A_n = \frac{\int_a^b f(x) g_n(x) dx}{\int_a^b g_n^2(x) dx}$$

The properties of orthogonal functions may be used to solve equation 4.15 for C_n by formulating an infinite series for the appropriate form of $f(x)$. From equation 4.16 it is evident that we should choose $f(x)$= 1 and the orthogonal function $g_n(x)$= $sin(n\pi x/L)$. Substituting into equation 4.17 we obtain,

$$A_n = \frac{\int_0^L \sin\frac{n\pi x}{L} dx}{\int_0^L \sin^2\frac{n\pi x}{L} dx} = \frac{2}{\pi}\ \frac{(-1)^{n+1}+1}{n}$$

Hence from equation 4.16, we have

$$1 = \sum_{n=1}^{\infty} \frac{2}{\pi}\ \frac{(-1)^{n+1}+1}{n}\ \sin\frac{n\pi x}{L} \qquad (4\text{-}18)$$

which is simply the expansion of unity in a *Fourier series*. Comparing equations 4.15 and 4.18 we obtain

$$C_n = \frac{2[(-1)^{n+1}+1]}{n\pi \sinh(\frac{n\pi W}{L})} \qquad n= 1, 2, 3, \ldots. \qquad (4\text{-}19)$$

Substituting equation 4.19 into equation 4.14, we then obtain for the final solution

$$\theta_{(x,y)} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi x}{L}\frac{\sinh\frac{n\pi y}{L}}{\sinh\frac{n\pi W}{L}} \quad (4\text{-}20)$$

The above equation is a convergent series, from which the value of θ may be computed for any x and y. Representative results are shown in the form of isotherms for a schematic of the rectangular plate (see Figure 4.2).

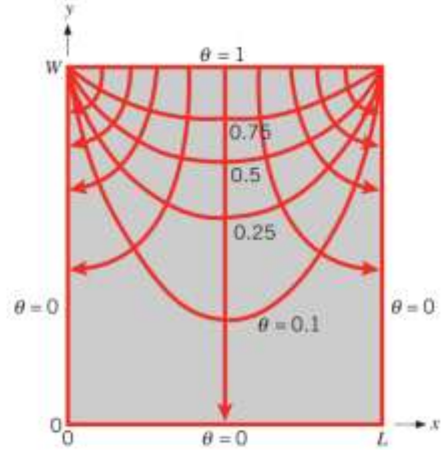


Figure 4.2: Isotherms and heat flow lines for two-dimensional conduction in a rectangular plate.

Example 2:

Derive an expression for the steady-state temperature distribution $\theta(x, y)$ of the extended surface as shown in Figure (4-3) for a finite heat transfer coefficient (h).

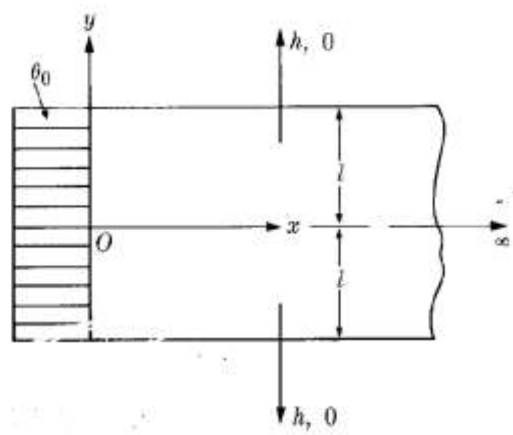


Figure 4.2: 2D extended surface.

__Solution:__

The transformed differential equation (applying Eq. 4.3) as,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

The boundary conditions are needed for each of the coordinates as,

$$\theta(0, y) = \theta_0\ , \quad \theta(\infty, y) = 0, \quad \frac{\partial\theta(x,0)}{\partial y} = 0\ , \quad -k\frac{\partial\theta(x,l)}{\partial y} = h\theta(x, l)$$

Assume the existence of a product solution of the form (Eq. 4.5)

$$\theta_{(x,y)} = X_{(x)}\, Y_{(y)}$$

Where X is a function of x alone and Y is a function y. The solutions obtained by the separation of variables are in the form of a sum or integral, depending on whether the homogeneous direction is finite or extends to infinity, respectively.

$$\frac{\partial^2 X}{\partial x^2} - \lambda^2 X = 0 \qquad , \theta(0, y) = \theta_0 , \quad \theta(\infty, y) = 0$$

$$\frac{\partial^2 Y}{\partial y^2} + \lambda^2 Y = 0 \qquad , \frac{\partial \theta(x,0)}{\partial y} = 0 , \quad -k\frac{\partial \theta(x,l)}{\partial y} = h\theta(x, l)$$

The general form of the two-dimensional solution is,

$$\theta_{(x,y)} = \left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(C_3 \cos \lambda y + C_4 \sin \lambda y) \qquad (4\text{-}21)$$

Applying the condition that $\frac{\partial \theta(x,0)}{\partial y} = 0$, So

$$\frac{\partial \theta_{(x,y)}}{\partial y} = \left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(\lambda\, C_3 \sin \lambda y - \lambda\, C_4 \cos \lambda y) \qquad (4\text{-}22)$$

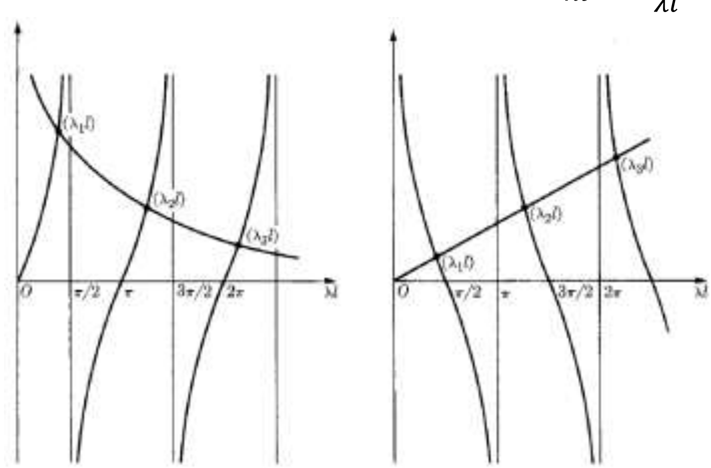
$$\frac{\partial \theta_{(x,0)}}{\partial y} = 0 = \left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(\lambda\, C_3 \sin(\lambda \times 0) - \lambda\, C_4 \cos(\lambda \times 0))$$

From this condition, it is evident that $C_4 = 0$. Therefore, the characteristic function is $(\cos \lambda y)$. In addition from the requirement that $-k\frac{\partial \theta(x,l)}{\partial y} = h\theta(x, l)$, we obtain

$$-k\left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(\lambda\, C_3 \sin(\lambda l)) = h\left(C_1 e^{-\lambda x} + C_2 e^{\lambda x}\right)(C_3 \cos \lambda l)$$

$\tan \lambda y = \frac{h}{k} \times \frac{1}{\lambda} \times \frac{l}{l} = \frac{1}{\lambda l} Bi$ or $\cot \lambda y = \frac{\lambda l}{Bi}$, where *Bi* is *Biot number* ($\frac{hl}{k} = Bi$)

The characteristic values are the roots of [$\tan \lambda_n y = \frac{Bi}{\lambda l}$ or $\cot \lambda_n y = \frac{\lambda_n l}{Bi}$] as shown,



Now, by applying the condition in *x*-axis ($\theta(\infty, y) = 0$) as

$\theta_{(\infty,y)} = \left(C_1 e^{-\lambda_n \infty} + C_2 e^{\lambda_n \infty}\right)(C_3 \cos \lambda_n y) = 0$, From this condition, it is evident that C_2= 0.

$$\theta_{(x,y)} = \sum_{n=1}^{\infty} A_n\, e^{-\lambda_n x} \cos \lambda_n y \qquad (4\text{-}23)$$

For last condition ($\theta(0, y) = \theta_0$),

$$\theta_0 = \sum_{n=1}^{\infty} A_n\, e^{(-\lambda_n \times 0)} \cos \lambda_n y$$

$$\theta_0 = \sum_{n=1}^{\infty} A_n \cos \lambda_n y \qquad (4\text{-}24)$$

Although the above equation (4-24) would seem to be an extremely complicated relation for evaluating A_n, a standard method is available. It involves writing an infinite series expansion in terms of *orthogonal functions*. An infinite set of functions $g_1(x)$, $g_2(x)$, … , $g_n(x)$, … is said to be orthogonal in the domain $a \le x \le b$ if, $\int_a^b g_m(x) g_n(x) dx = 0 \qquad m \neq n$

The form of the coefficients A_n in this series may be determined by multiplying each side of the equation by $g_m(x)$ and integrating between the limits *a* and *b*.

$$\int_a^b f(x) g_m(x) dx = \int_a^b g_m(x) \sum_{n=1}^{\infty} A_n g_n(x)\, dx$$

However, from above equation it is evident that all but one of the terms on the *right-hand* side of equation 4.16 must be zero, leaving us with

$$\int_a^b f(x) g_m(x) dx = A_m \int_a^b g_m^2\,(x) dx \qquad (m = n)$$

Therefore, $\theta_0 \int_0^l \cos \lambda_n y\, dy = A_n \int_0^l \cos^2 \lambda_n y dy$

$$\frac{\theta_0}{\lambda_n} [\sin \lambda_n y]_0^l = \frac{A_n}{\lambda_n} \left[\frac{\lambda_n y}{2} + \frac{1}{4} \sin 2\lambda_n y\right]_0^l$$

$$2\theta_0 \sin \lambda_n l = A_n [\lambda_n l + \sin \lambda_n l \cos \lambda_n l]$$

$$A_n = \frac{2\theta_0 \sin \lambda_n l}{\lambda_n l + \sin \lambda_n l \cos \lambda_n l}$$

Hence, the steady-state temperature distribution $\theta(x, y)$ of the extended surface is,

$$\frac{\theta_{(x,y)}}{\theta_0} = 2 \sum_{n=1}^{\infty} \left(\frac{\sin \lambda_n l}{\lambda_n l + \sin \lambda_n l \cos \lambda_n l}\right) e^{-\lambda_n x} \cos \lambda_n y$$

4.7 Nonhomogeneous Problems

There are many engineering problems in which the P.D.Es. and/ or the B.Cs. are not homogeneous. An example would be a plane-wall, nuclear reactor fuel element which is suddenly turned on. The P.D.E. describing this problem is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (4\text{-}25)$$

The generation term $(\frac{\dot{g}}{k})$ makes the above equation non-homogenous. Another example would be a plane wall subjected to an ambient fluid whose temperature is fluctuating with time, defined as;

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}; \qquad \frac{\partial T(0,t)}{\partial x} = 0 \quad , T(L,t) = A\sin(\omega t)$$

In this case, the P.D.E. is homogeneous as is the B.C. at (x= 0). The B.C. at (x= L) is non-homogeneous (but linear), and consequently the problem is non-homogeneous, (but linear). The above examples cannot be made homogeneous by simply subtracting a constant from (T) $[\theta = T - Const.]$ as can be done in the lumped heat capacity problem. This section will discuss two methods of handing these more complicated problems;

- ✓ Partial Solutions.
- ✓ Variation of Parameters.

✓ Partial Solutions

A non-homogeneous problem can often be converted into a homogeneous one by the use of "*Partial Solution*" to the nonhomogeneous problem. A partial solution is one that satisfies only a part of the original problem. In a transient problem the most common partial solution would be the steady state solution. It satisfies the B.Cs. but not the initial condition. In addition, it deletes the time derivative from the P.D.E.

The steps in obtaining solutions to non-homogeneous problems by using the steady-state solution are;

1. Let $\theta = \theta_h + \theta_p$ where θ the general solution, θ_h is homogenous solution and θ_p is partial solution.
2. Determine the steady-state solutionθ_h.
3. Introduce ($\theta_p = \theta - \theta_h$) to make the problem homogeneous.
4. Solve for (θ_p) in the usual manner (*Separation of Variables*).
5. The complete solution is ($\theta = \theta_h + \theta_p$).

Example 3:

Consider an electric heater made from a solid rod of rectangular cross section (2*L*×2*l*) and designed according to one of the forms shown in **Figure 4.3**. The temperature variation along the rod can be neglected. The internal energy generation ($\dot{g}$) in the heater is uniform. The heat transfer coefficient (h) is large. Find the steady-state temperature of the electric heater.

Solution:

The formulation of the problem in Figure 4.3 is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\dot{g}}{k} = 0 \qquad (4\text{-}26)$$

The Boundary conditions are,

$$\frac{\partial\theta(0,y)}{\partial x} = 0 \qquad \frac{\partial\theta(x,0)}{\partial y} = 0$$

$$\theta(L,y) = 0 \qquad \theta(x,l) = 0$$

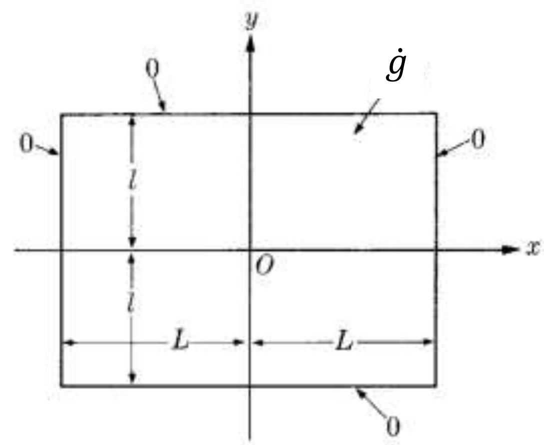


Figure 4.3: electric heater with rectangular cross section area.

The above partial differential equation, being *Non-homogenous*, is not separable.

Therefore, the general solution of the problem is now assumed to be,

$$\theta(x,y) = \theta_h(x,y) + \theta_p(x) \qquad (4\text{-}27)$$

Or, $\theta(x,y) = \theta_h(x,y) + \theta_p(y)$ (4-28)

$\theta_h(x,y)$ is solution of homogeneous part of the partial differential equation $[\nabla^2\theta_h = 0]$ with neglecting ($\dot{g}$).

$\theta_p(x)$ or $\theta_p(y)$ is partial solution part of the partial differential equation $[\frac{\partial^2\theta_p}{\partial x^2} + \frac{\dot{g}}{k} = 0]$ or $[\frac{\partial^2\theta_p}{\partial y^2} + \frac{\dot{g}}{k} = 0]$.

With the inclusion of the internal energy generation ($\dot{g}$) in the formulation of the One-dimensional problem, $\theta_p(x)$ or $\theta_p(y)$, the differential equation to be satisfied by the Two-dimensional problem, $\theta_h(x,y)$, can be made homogeneous.

Partial solution part:

$$\frac{\partial^2\theta_p}{\partial x^2} + \frac{\dot{g}}{k} = 0 \qquad \longrightarrow \qquad \theta_p = \frac{\dot{g}}{2k}x^2 + Ax + B$$

B.C.1: $\frac{\partial\theta_p(x=0)}{\partial x} = 0 \qquad \longrightarrow \qquad A = 0$

B.C.2: $\theta(x = L) = 0 \qquad \longrightarrow \qquad B = \frac{\dot{g}}{2k}L^2$

$$\theta_p = \frac{\dot{g}}{2k}L^2\left(1 - \left(\frac{x}{L}\right)^2\right) \tag{4-29}$$

The homogeneous solution part:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0 \tag{4-30}$$

B.C.1: $\frac{\partial\theta_h(0,y)}{\partial x} = 0$

B.C.2: $\theta_h(L,y) = 0$

B.C.3: $\frac{\partial\theta_h(x,0)}{\partial y} = 0$

B.C.4: $\theta_h(x,l) = -\theta_p(x)$

From Eq. 4.27, $\theta(x,y) = \theta_h(x,y) + \theta_p(x)$,

$$\theta_h(x,y) = \theta(x,y) - \theta_p(x) \qquad \longrightarrow \qquad \frac{\partial\theta_h(x,y)}{\partial y} = \frac{\partial\theta(x,y)}{\partial y} - \frac{\partial\theta_p(x)}{\partial y} = 0$$

Now let, $\theta_h(x,y) = X(x).Y(y)$ (4-31)

Applying Eq. 4.31 into Eq. 4.30,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2$$

Therefore, $\theta_h(x,y) = (C_1 \cos\lambda x + C_2 \sin\lambda x)(C_3 \cosh\lambda y + C_4 \sinh\lambda y)$ (4-32)

From B.C.1, $C_2 = 0$ & From B.C.3, $C_4 = 0$

Thus, $\theta_h(x,y) = C_1 \cos\lambda x\, C_3 \cosh\lambda y$ (4-33)

From B.C.2: $0 = C_1 \cos\lambda L\, C_3 \cosh\lambda y$ $0 = C \cos\lambda L \cosh\lambda y$

Since $\cosh\lambda y \neq 0$ $\cos\lambda L = 0$ $\lambda L = \frac{\pi}{2}(2n+1)$ (4-34)

$$\lambda_n = \frac{\pi}{2}\frac{(2n+1)}{L}$$

Thus, $\theta_h(x,y) = \sum_{n=0}^{\infty} C_n \cos\lambda_n x \cosh\lambda_n y$ (4-35)

From B.C.4: $\sum_{n=0}^{\infty} C_n \cos\lambda_n x \cosh\lambda_n l = -\frac{\dot{g}}{2k}L^2\left(1-\left(\frac{x}{L}\right)^2\right)$ $\times (\cos\lambda_n x)$

$$C_n \cosh\lambda_n l \int_0^L \cos^2\lambda_n x\, dx = -\int_0^L \frac{\dot{g}}{2k}L^2\left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx$$

$$C_n \cosh\lambda_n l \frac{L}{2} = -\int_0^L \frac{\dot{g}}{2k}L^2\left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx$$

$$C_n = \frac{-1}{\cosh\lambda_n l}\frac{2}{L}\int_0^L \frac{\dot{g}}{2k}L^2\left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx \quad \rightarrow$$

$$C_n = \frac{-1}{\cosh\lambda_n l}\frac{\dot{g}L}{k}\int_0^L \left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx$$

Now, taken: $\int_0^L \left(1-\left(\frac{x}{L}\right)^2\right)\cos\lambda_n x\, dx = \int_0^L \cos\lambda_n x\, dx - \frac{1}{b^2}\int_0^L x^2 \cos\lambda_n x\, dx$

Integrating left hand side, $\int_0^L \cos\lambda_n x\, dx = \frac{1}{\lambda_n}\sin\lambda_n L = \frac{1}{\lambda_n}\sin\frac{2n+1}{L}\frac{\pi}{2}L = \frac{1}{\lambda_n}(-1)^n$

Integrating right hand side, $\int_0^L \underbrace{x^2}_{u}\underbrace{\cos\lambda_n x\, dx}_{dv} = x^2\frac{\sin\lambda_n x}{\lambda_n}\Big]_0^L - 2\int_0^L \frac{\sin\lambda_n x dx}{\lambda_n}$

$$= \frac{x^2 \sin\lambda_n x}{\lambda_n}\Big]_0^L - \frac{2}{\lambda_n}\left(x\frac{-\cos\lambda_n x}{\lambda_n} + \int_0^L \frac{\cos\lambda_n x dx}{\lambda_n}\right)$$

$$= \frac{x^2 \sin\lambda_n x}{\lambda_n}\Big]_0^L + \frac{2}{\lambda_n}x\cos\lambda_n x\Big|_0^L - \frac{2}{\lambda_n{}^3}[\sin\lambda_n x]_0^L$$

$= \frac{x^2 \sin \lambda_n x}{\lambda_n}\Big|_0^L + \frac{2}{{\lambda_n}^2} x\cos \lambda_n x\Big|_0^L - \frac{2}{{\lambda_n}^3} \sin \lambda_n x\Big|_0^L$

$= \frac{L^2 \sin \lambda_n L}{\lambda_n} + \frac{2}{{\lambda_n}^2} L \cos \lambda_n L - \frac{2}{{\lambda_n}^3} \sin \lambda_n L = \frac{L^2}{\lambda_n}(-1)^n + \frac{2L}{{\lambda_n}^2}(0) - \frac{2}{{\lambda_n}^3}(-1)^n$

$= (-1)^n (\frac{L^2}{\lambda_n} - \frac{2}{{\lambda_n}^3})$

So, $\int_0^L \left(1 - \left(\frac{x}{L}\right)^2\right) \cos \lambda_n x \, dx = \frac{(-1)^n}{\lambda_n} - \frac{(-1)^n}{L^2}(\frac{L^2}{\lambda_n} - \frac{2}{{\lambda_n}^3}) = \frac{2}{L^2 {\lambda_n}^3}(-1)^n$

$C_n = \frac{-1}{\cosh \lambda_n l} \frac{\dot{g}L}{k} \frac{2}{L^2 {\lambda_n}^3}(-1)^n = \frac{-2\dot{g}}{kL^2 {\lambda_n}^3 \cosh \lambda_n l}(-1)^n$

$$\theta(x,y) = \sum_{n=0}^{\infty} \frac{-2\dot{g}}{kL^2 {\lambda_n}^3 \cosh \lambda_n l}(-1)^n \cos \lambda_n x \cosh \lambda_n y + \frac{\dot{g}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

$\frac{\theta(x,y)}{\frac{\dot{g}L^2}{2k}} = \frac{1}{2}\left(1 - \left(\frac{x}{L}\right)^2\right) - 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n L)^3} \frac{\cosh \lambda_n y}{\cosh \lambda_n l} \cos \lambda_n x$

- ✓ **Variation of Parameters**

In some cases there may not be steady-state solution or you may not be able to find it. In these cases; variation of parameters method may be used. The procedure is outlined as follows;

1. Set up a problem corresponding to the original one by simply setting all the non-homogeneous terms equal to zero.

2. Determine the *eigen-functions* and *eigen-condition* for the "corresponding homogeneous problem".

3. Construct a solution to the original non-homogeneous problem of the form;

$\theta(x,t) = \sum_n A_n(t)\emptyset_n(x)$

Where, the $\emptyset_n(x)$ is the eigen-functions you have obtained above from the corresponding homogeneous problem.

4. Evaluate $A_n(t)$ in the usual manner making use of the orthogonally of the $\emptyset_n(x)$.

That is; $\int_0^1 \theta(x,t)\, \varphi_m(x) dx = \sum_n^{\infty} A_n(t) \int_0^1 \varphi_n(x)\varphi_m(x) dx = CA_m(t)$

Or, $A_m(t) = \frac{1}{C}\int_0^1 \theta(x,t)\,\varphi_m(x)dx$

Here θ is still unknown.

5. Set up an O.D.E. and B.Cs. for $A_m(t)$.

6. Solve for $A_m(t)$.

7. Complete the solution; $\theta(x,t) = \sum_n A_n(t)\emptyset_n(x)$

Example 4:

Consider a plane wall whose initial normalized temperature is zero. The face (*x*= 0) is suddenly changed to a normalized temperature of unity at zero time, while the face at (*x*= 1) is maintained at the initial temperature. Determine the temperature distribution.

Solution:

(1) Set every non-homogeneous term equal to zero, thus the problem will be;

$$\frac{\partial^2\theta_h}{\partial x^2} = \frac{\partial\theta_h}{\partial t} \;\ldots\ldots\ldots(1)$$

$$\left.\begin{aligned} &\text{I.C.} \quad \theta_h(x,0) = 0 \\ &\text{B.C.1} \quad \theta_h(0,t) = 0 \\ &\text{B.C.2} \quad \theta_h(1,t) = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots(2)$$

(2) Determine the eigen-functions.

$$\text{Let } \theta_n = \mathrm{X(x)}\tau(\mathrm{t}) \qquad \ldots\ldots(3)$$

$$\text{Thus;} \quad \frac{X''}{x} = \frac{\tau'}{\tau} = -\lambda^2$$

$$\left.\begin{aligned} &X'' + \lambda^2 \mathrm{X} = o \\ &\mathrm{X}(0) = 0 \\ &\mathrm{X}(1) = 0 \end{aligned}\right\} \begin{aligned} &\mathrm{X = A\,sin\lambda x + B\,cos\lambda x} \\ &\mathrm{X}(0) = 0 \quad \rightarrow B = o \\ &\mathrm{X}(1) = 0 \quad \rightarrow \sin\lambda = o\,, \lambda_n = n\pi \end{aligned}$$

Therefore;

$\Phi_n(\mathrm{x}) = \sin(\mathrm{n}\pi x)$ ……(4)

(eigen-function corresponds to the homogenous problem)

3- Construct a solution;

$$\theta(\mathrm{x,t}) = \sum_{n=1}^{\infty} A_n(\mathrm{t})\,\Phi_n(\mathrm{x}) = \sum_{n=1}^{\infty} A_n(\mathrm{t}) \sin(\mathrm{n}\pi x) \ldots\ldots (5)$$

4- $\int_o^1 \theta(x,t) sinm\pi x dx = \sum_{n=1}^{\infty} A_n(\mathrm{t}) \int_o^1 sinn\pi x\; sinm\pi x dx = \frac{1}{2} A_m(\mathrm{t})$

$$where; \quad A_m(\mathrm{t}) = 2\int_o^1 \theta(x,t)\sin(m\pi x)\,dx \ldots\ldots (6)$$

5- Set up an O.D.E. for $A_m(t)$ by differentiating with respect to (t). Thus;

$$\frac{dA_m(\mathrm{t})}{dt} = 2\int_o^1 \frac{\partial\theta(x,t)}{\partial t}\sin(\mathrm{m}\pi x)dx$$

Now, since; $\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t} \Longrightarrow \frac{\partial\theta(x,t)}{\partial t} = \frac{\partial^2\theta(x,t)}{\partial x^2}$, hence;

$$\frac{dA_m(\mathrm{t})}{dt} = 2\int_o^1 \frac{\partial^2\theta}{\partial x^2}\sin(\mathrm{m}\pi x)dx = 2\left\{\left(\frac{\partial\theta}{\partial x}\sin m\pi x\right)_o^1 - m\pi\int_o^1 \frac{\partial\theta}{\partial x}\cos m\pi x dx\right\}$$

Noting that; $\frac{\partial\theta}{\partial x}(1,\mathrm{t})$ is unknown, but $\sin \mathrm{m}\pi 1 = o\; for x = 1$.

Similarly $\frac{\partial\theta}{\partial x}(0,\mathrm{t})$ is unknown, but $\sin \mathrm{m}\pi 0 = 0\; for\; x = 0$.

Thus; $\frac{dA_m(\mathrm{t})}{dt} = -2m\pi\int_o^1 \frac{\partial\theta}{\partial x}\cos m\pi x dx$

$$= -2\mathrm{m}\pi\left\{[\theta\cos m\pi]_0^1 + m\pi\int_0^1 \theta\, sinm\pi x dx\right\}$$

$$\frac{dA_m(\mathrm{t})}{dt} = -2m\pi\left\{\theta(1,t)\cos m\pi - \theta(o,t)\,1 + m\pi\frac{A_m(\mathrm{t})}{2}\right\}$$

$$\frac{dA_m(\mathrm{t})}{dt} = 2m\pi - m^2\pi^2 A_m(t)$$

$$\frac{dA_m}{dt} + m^2\pi^2 A_m = 2m\pi \ldots\ldots (7)$$

I.C. $A_m(0) = 2\int_o^1 \theta(x,0)\sin m\pi x dx \rightarrow A_m(0) = 0$

Equation (7) can be solved using integrating function (I);

$$\mathrm{I} = \exp \int m^2 \, \pi^2 \; dt = e^{m^2 \pi^2 t}$$

Then; $\frac{d}{dt}(A_m e^{m^2\pi^2 t}) = 2\mathrm{m}\pi e^{m^2\pi^2 t}$

$$A_m e^{m^2\pi^2 t} = \frac{2m\pi}{m^2\pi^2} \, e^{m^2\pi^2 t} \; + C_1$$

$$A_m(0) = 0 \; \rightarrow C_1 = -\frac{2}{m\pi}$$

Thus; $A_m(\mathrm{t}) = \frac{2}{m\pi} - \frac{2}{m\pi} \, e^{-m^2\pi^2 t}$

Hence, from (5) $\theta(x,t) = \sum_{n=1}^{\infty}(\frac{2}{n\pi} - \frac{2}{n\pi} \, e^{-n^2\pi^2 t})\sin \mathrm{n}\pi x$

Or; $\theta(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin \pi x}{n} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{sin\pi x e^{-n^2\pi^2 t}}{n}$

4.8 Cylindrical Geometry

The inherent nature of cylindrical coordinates implies three types of 2-D problems in the form $T(r,\varphi)$, $T(r,z)$ and $T(\varphi,z)$, see Figure 4.4.

Since $T(\varphi,z)$ has no physical significance (except in thin–walled tubes, which can be investigated in terms of Cartesian Coordinates), it is not considered here. $T(r,z)$ may depend on the expansion of an arbitrary function into a series in terms of cylindrical (*Bessel*) functions. Problems of the type $T(r,\varphi)$, on the other hand, require no further mathematical background than that needed for Cartesian geometry.

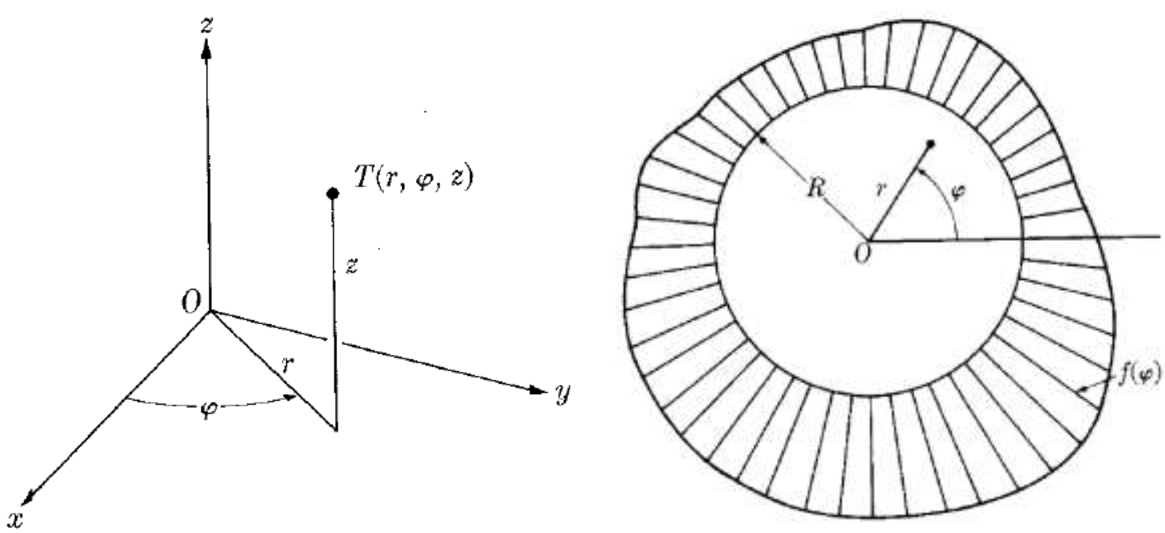


Figure 4.4: Infinitely long rod with cylindrical coordinates.

When a problem of type $T(r,z)$ is orthogonal in the (r-direction), it can be solved by the proper choice of the separation constant leading to a 2^{nd} order O.D.E. in (z) satisfied by hyperbolic function, and to Bessel equation in (r). If the z-direction is orthogonal, the problem do not need additional mathematics and can be solved by using circular functions in Z and the *modified Bessel* functions in (r).

Example 5:

The surface temperature of an infinitely long solid rod of radius R is specified as $f(\varphi)$, see Figure 4.4. Find the steady-state temperature of the rod.

Solution:

For steady-state $\frac{\partial}{\partial \mathrm{t}} = 0$, No heat generation $\dot{g} = 0$, 2D $\frac{\partial}{\partial \mathrm{z}} = 0$

Thus: $\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial \mathrm{T}}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2}\left(\frac{\partial^2 \mathrm{T}}{\partial \phi^2}\right) = 0$ (4-36)

$$\left(\frac{\partial^2 T}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 T}{\partial \phi^2}\right) = 0$$

And the problem boundary condition as:

$T(0,\varphi) = finite$, $T(R,\varphi) = f(\varphi)$,

$T(r,\varphi) = T(r,\varphi + 2\pi)$, $\frac{\partial T(r,\varphi)}{r\partial\varphi} = \frac{\partial T(r,\varphi+2\pi)}{r\partial\varphi}$

The r-direction cannot be made orthogonal by any transformation. This leaves (φ) as the only possible orthogonal direction. Hence the product solution $T(r,\varphi) = R(r)\phi(\varphi)$ with the proper choice of the separation constant yields

$T(r,\varphi) = R(r)\phi(\varphi)$ (4-37)

By substituting Equation 4.37 into Equation 4.36 to obtain that

$\left(\frac{\partial^2 R(r)}{\partial r^2}\phi(\varphi)\right) + \frac{1}{r}\left(\frac{\partial R(r)}{\partial r}\phi(\varphi)\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}R(r)\right) = 0$ Dividing by $(R(r)\phi(\varphi))$

$$\left(\frac{\partial^2 R(r)}{\partial r^2}/R(r)\right) + \frac{1}{r}\left(\frac{\partial R(r)}{\partial r}/R(r)\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}/\phi(\varphi)\right) = 0$$

$$r^2\left(\frac{\partial^2 R(r)}{\partial r^2}/R(r)\right) + r\left(\frac{\partial R(r)}{\partial r}/R(r)\right) = -\left(\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}/\phi(\varphi)\right) = \lambda^2$$

$\frac{\partial^2 \phi(\varphi)}{\partial \varphi^2} + \lambda^2\phi(\varphi) = 0$ with B.Cs. $T(r,\varphi) = T(r,\varphi + 2\pi)$ & $\frac{\partial T(r,\varphi)}{r\partial\varphi} = \frac{\partial T(r,\varphi+2\pi)}{r\partial\varphi}$

$r^2\left(\frac{\partial^2 R(r)}{\partial r^2}\right) + r\left(\frac{\partial R(r)}{\partial r}\right) - \lambda^2 R(r) = 0$ with B.Cs. $T(0,\varphi) = finite$

$\phi(\varphi) = A\cos\lambda\varphi + B\sin\lambda\varphi$ (4-38)

$$A\sin\lambda\varphi + B\cos\lambda\varphi = A\sin\lambda(\varphi + 2\pi) + B\cos\lambda(\varphi + 2\pi)$$

But, $\sin\lambda\varphi = \sin\lambda(\varphi + 2\pi) = \sin\lambda\varphi\cos 2\lambda\pi + \cos\lambda\varphi\sin 2\lambda\pi$

$\cos\lambda\varphi = \cos\lambda(\varphi + 2\pi)$ where $\lambda = n,$ $n = 0, 1, 2, \ldots..$

$$\phi(\varphi) = A_n \cos(n\varphi) + B_n \sin(n\varphi) \tag{4-39}$$

Sol. of (5) $r^2R'' + rR' - \lambda^2 R = 0$

Let; $r = e^z \rightarrow z = \ln r$

$$R' = \frac{dR}{dr} = \frac{dR}{dz}\frac{dz}{dr} = \frac{1}{r}\frac{dR}{dz}$$

$$\frac{d^2R}{dr^2} = R'' = \frac{d}{dr}\left(\frac{dR}{dz}\right) = -\frac{1}{r^2}\frac{dR}{dz} + \frac{1}{r^2}\frac{d^2R}{dz^2}$$

Thus; $\frac{d^2R}{dz^2} - \frac{dR}{dz} + \frac{dR}{dz} - \lambda^2 R = 0$

Or; $\frac{d^2R}{dz^2} - \lambda^2 R = 0$

Thus; $R = c\,e^{\lambda z} + De^{-\lambda z} = ce^{\lambda lnr} + De^{-\lambda lnr}$

Or; $R = c\,r^{\lambda} + Dr^{-\lambda}$ ……. (7)

Or, $(\lambda = n)$;

$R = c_n r^n + D_n r^{-n}$

B.C.1 R(0) = finite $\rightarrow D_n = 0$

Hence; $R_n = C_n r^n$ …… (8)

Thus; $T(r,\phi) = \sum_{n=o}^{\infty} R_n\,\Phi_n = R_0\,\Phi_0 + \sum_{n=1}^{\infty} R_n\,\phi_n$

Hence; $T(r,\phi) = a_0 + \sum_{n=1}^{\infty} r^n\,(a_n \cos(n\phi) + b_n \sin(n\phi))$ ..(9)

Where; $a_0 = A_0\,C_0$, $a_n = A_n\,C_n$, $b_n = B_n\,C_n$

Using B.C.2;

$f(\phi) = a_0 + \sum_{n=1}^{\infty} R^n\,(a_n \cos n\phi + b_n \sin n\phi)$ ……. (10)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi$$

$$a_n R_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi$$

$$b_n R^n = \frac{1}{\pi}\int_o^{2\pi} f(\phi) sinn\phi d\phi$$

For the problem defined by;

$$T(R,\phi) = T_o \quad 0 < \phi < \pi$$

$$= 0 \quad \pi < \phi < 2\pi$$

Equ.(11) gives;

$$\left.\begin{aligned} a_0 &= \frac{1}{2} T_0 \\ a_n &= 0 \\ b_n R^n &= 2T_0 / n\pi \end{aligned}\right\} n=1,3,5....$$

Thus;

$$\frac{T(r,\phi)}{T_0} = \frac{1}{2} + 2 \sum_{n=1}^{\infty} \frac{1}{n\pi} \left(\frac{r}{R}\right)^n \sin(n\phi)$$

4.9 Spherical Geometry

When a spherical problem depends on the cone angle (θ), see the figure, its solution can be reduced to the expansion of arbitrary function into series of "*Legendre Polynomial*". The linear 2nd order differential equation with variable coefficients.

The linear second-order differential equation with variable coefficients

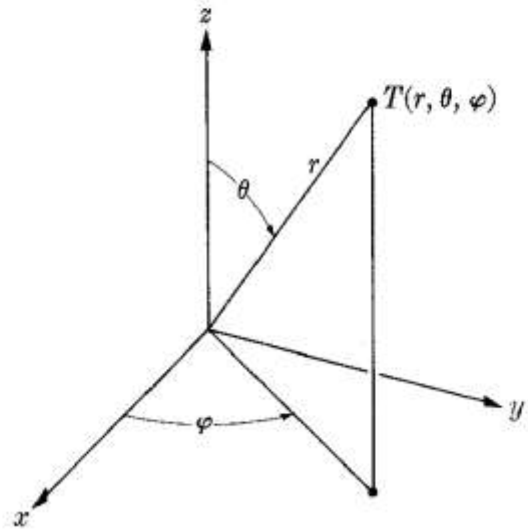


Figure 4.5: Spherical coordinates.

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \qquad (4\text{-}40)$$

Is known as "*Legendre's Equation*", and its solutions are known as "*Legendre*'s *Functions*". In particular, when (*n*= 0) or a positive integer, the solutions of (4-40) are called "*Legendre Polynomial*". The solution of (4-40) may be obtained by the method of power series as;

$$y(x)=a_0P_n(x)+a_1Q_n(x) \qquad (4\text{-}41)$$

Where,

$P_n(x)$ is *Legendre Polynomial* of degree (n) of the first kind.

$Q_n(x)$ is *Legendre Polynomial* of degree (n) of the second kind.

Hence the *Legendre Polynomials* $P_n(x)$ are the characteristic functions of the characteristic value problem stated by [$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+n(n-1)y=0$].

These polynomials form an orthogonal set with respect to the weighting function $w(x)=1$ over the interval (-1, 1); that is

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx=0 \qquad \text{if} \qquad m\neq n. \qquad (4\text{-}42)$$

Now the expansion of an arbitrary function $f(x)$ in terms of appropriate Legendre polynomials, the *Fourier-Legendre series*, may be written in the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \qquad -1 < x < 1$$

Here the coefficient a_n again follows

$$a_n = \frac{\int_{-1}^{1} f(x) P_n(x)\, dx}{\int_{-1}^{1} P_n^2(x)\, dx} \tag{4-43}$$

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2 - 1)^n\, dx \tag{4-44}$$

If $f(x)$ and its first n derivatives are continuous in the interval, integrating the right-hand side of the above equation n times by parts gives

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} (x^2 - 1)^n \frac{d^n f(x)}{dx^n}\, dx \tag{4-45}$$

Now, replacing by in the above equation, and employing the n^{th} derivative of equation [$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$. $\qquad \frac{d^n P_n(x)}{dx^n} = \frac{(2n)!}{2^n n!}$

we find the denominator to be

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{(2n)!}{2^{2n} (n!)^2} \int_{-1}^{1} (1 - x^2)^n\, dx \tag{4-46}$$

The right-hand side of the above equation integrated n times by parts yields

$$\int_{-1}^{1} (1 - x^2)^n\, dx = \frac{2^{2n+1} (n!)^2}{(2n + 1)!} \tag{4-47}$$

Introducing Equation (4-47) into Equation (4-46), obtain

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n + 1}. \tag{4-48}$$

Hence the coefficient (a_n) becomes,

$$a_n = \begin{cases} \dfrac{2n + 1}{2} \displaystyle\int_{-1}^{1} f(x) P_n(x)\, dx, \\[2ex] \dfrac{2n + 1}{2^{n+1} n!} \displaystyle\int_{-1}^{1} (1 - x^2)^n \frac{d^n f(x)}{dx^n}\, dx. \end{cases} \tag{4-49}$$

The second form of Equation (4.49) can be used only if $f(x)$ and its first n derivatives are continuous in (-1, 1).

Furthermore, noting that $P_n(x)$ is an even function of x when n is ***even***, and an odd function when n is ***odd***, we have

For an ***Even*** function $f(x)$,

$$a_n = \begin{cases} (2n+1)\int_0^1 f(x)P_n(x)\,dx, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \qquad (4\text{-}50)$$

For an ***Odd*** function $f(x)$

$$a_n = \begin{cases} (2n+1)\int_0^1 f(x)P_n(x)\,dx, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \qquad (4\text{-}51)$$

<u>Example 6:</u>

The surface temperature of a sphere of radius R is specified in the form $f(\theta)$. Find the steady-state temperature distribution of the sphere.

<u>Solution:</u>

The formulation of the problem

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = 0,$$

$$T(0,\theta) = \text{finite},$$

$$T(R,\theta) = f(\theta).$$

The missing boundaries in the θ-direction will be discussed later.

Since θ is the only possible orthogonal direction, with the appropriate choice of separation constant the product solution $[T(r,\theta) = \mathcal{R}(r)\vartheta(\theta)]$ yields

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\vartheta}{d\theta}\right) + \lambda\vartheta = 0, \quad \text{and} \quad r^2\frac{d^2\mathcal{R}}{dr^2} + 2r\frac{d\mathcal{R}}{dr} - \lambda\mathcal{R} = 0$$

First rearranged in the form
$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{1-\cos^2\theta}{\sin\theta}\frac{d\vartheta}{d\theta}\right)+\lambda\vartheta=0,$$

Then transformed with $x = \cos\theta \rightarrow [\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=\sin\theta\frac{d}{dx}]$ and $[\frac{d\vartheta}{d\theta}=\frac{d\vartheta}{dx}\frac{dx}{d\theta}=\sin\theta\frac{d\vartheta}{dx}]$, may be written as,
$$\frac{d}{dx}\left[(1-x^2)\frac{d\vartheta}{dx}\right]+n(n+1)\vartheta=0,$$

where $n(n+1)=\lambda$. This is a Legendre equation. Its particular solution, finite at $x=\pm1(\theta=0,\pi)$, is

$$\vartheta_n = A_n\psi_n(\theta), \qquad \psi_n(\theta)=P_n(\cos\theta), \quad \text{characteristic functions,}$$

$$n = 0, 1, 2, 3, \ldots, \quad \text{characteristic values.}$$

Here the condition of finiteness specifying the characteristic functions and characteristic values takes care of the two missing boundary conditions in the θ-direction. The general solution of the equidimensional equation given by:

$$\mathcal{R}_n(r) = C_n r^n + D_n r^{-(n+1)} \qquad \text{where } n=-\tfrac{1}{2}+(\lambda+\tfrac{1}{4})^{1/2}.$$

Thus, $\mathcal{R}_n(r) = C_n r^n$.

Thus the product solution of the problem is

$$T(r,\theta)=\sum_{n=0}^{\infty}a_n r^n P_n(\cos\theta), \qquad \text{where } a_n = A_nC_n.$$

The use of $T(R,\theta)=f(\theta)$ and reduced the above equation to

$$f(\theta)=\sum_{n=0}^{\infty}a_nR^nP_n(\cos\theta),$$

which is the expansion of $f(\theta)$ into a Fourier-Legendre series. Here the coefficient a_n is readily obtained

$$a_nR^n=\frac{2n+1}{2}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta\,d\theta$$

In particular, if the surface temperature is specified† as

$$T(R, \theta) = f(\theta) = \begin{cases} T_0, & 0 < \theta < \pi/2, \\ 0\,, & \pi/2 < \theta < \pi, \end{cases}$$

$$a_n R^n = T_0 \left(\frac{2n+1}{2}\right) \int_0^1 P_n(x)\, dx$$

$$a_0 = \tfrac{1}{2} T_0 \int_0^1 dx = \tfrac{1}{2} T_0,$$

$$a_1 R = \tfrac{3}{2} T_0 \int_0^1 x\, dx = \tfrac{3}{4} T_0,$$

$$a_2 R^2 = 0,$$

$$a_3 R^3 = \tfrac{7}{2} \cdot \tfrac{1}{2} T_0 \int_0^1 (5x^3 - 3x)\, dx = -\tfrac{7}{16} T_0,$$

$$a_4 R^4 = 0,$$

$$a_5 R^5 = \tfrac{11}{2} \cdot \tfrac{1}{8} T_0 \int_0^1 (63x^5 - 70x^3 + 15x)\, dx = \tfrac{11}{32} T_0,$$

$$\vdots$$

Hence the solution of the problem is

$$\frac{T(r, \theta)}{T_0} = \frac{1}{2} + \frac{3}{4}\left(\frac{r}{R}\right) P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) + \frac{11}{32}\left(\frac{r}{R}\right)^5 P_5(\cos\theta) + \cdots$$

4.10 Heterogeneous Solids (Variable Thermal Conductivity)

Heterogeneous solids are becoming increasingly important because of the large ranges of temperature involved in problems of technology, as in reactor fuel elements, space vehicle components, solidification of castings …etc. The equation of heat conduction for heterogeneous solids;

Cartesian coordinates

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{g}=\rho C_p\frac{\partial T}{\partial t}$$

Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{g}=\rho C_p\frac{\partial T}{\partial t}$$

Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{g}=\rho C_p\frac{\partial T}{\partial t}$$

If k, C_p, $\dot{g}$ are functions of space only, the above three equations become a linear differential equation with variable coefficients. If (k & C_p) are dependent on temperature but independent of space, however, the above equations become nonlinear and difficult to solve. Usually numerical methods have to be employed. A number of analytical methods are also available. One of these, *Kirchhoff's method*, is to a large extent general.

Above equations may be reduced to a linear differential equation by introducing a new temperature θ related to the temperature T of the problem by the *Kirchhoff transformation*,

$$\theta=\frac{1}{k_R}\int_{T_R}^{T}k(T)\,dT \qquad (4\text{-}52)$$

where T_R denotes a convenient reference temperature, and $k_R = k(T_R)$. T_R and k_R are introduced merely to give θ the dimensions of temperature and a definite value. It follows from Eq. (4-52) that

$$\frac{d\theta}{dt} = \frac{k}{k_R}\frac{dT}{dt} \tag{4-53}$$

$$\nabla\theta = \frac{k}{k_R}\,\nabla T \tag{4-54}$$

Inserting Eqs. (4-53) and (4-54) into energy equations, we have

$$\frac{d\theta}{dt} = a\nabla^2\theta + \left(\frac{a}{k_R}\right)\dot{g} \tag{4-55}$$

where a and $\dot{g}$ are expressed as functions of the new variable θ. For many solids, however, the temperature dependence of a can be neglected compared to that of k. In such cases, if $\dot{g}$ is independent of T, Eq. (4-55) becomes identical to Eq. (4-52) except for the different but constant coefficient of $\dot{g}$. Thus the solutions obtained for homogeneous solids may be readily utilized for heterogeneous solids by replacing T by θ and ρC_p by k_R/a, provided that the boundary conditions prescribe T or $\frac{\partial T}{\partial n}$. This remark does not hold if the boundary conditions involve the convective term $h(T_o - T\infty)$. The following one-dimensional example illustrates the use of the method.

Example 7:

A liquid is boiled by a flat electric heater plate of thickness $2L$. The internal energy $\dot{g}$ generated electrically may be assumed to be uniform. The boiling temperature of the liquid, corresponding to a specified pressure, is Too (see Fig. 4.6). Find the steady-state temperature of the plate for

(i) $k = k(T)$; (ii) $k = k_R$ (l +(βT).

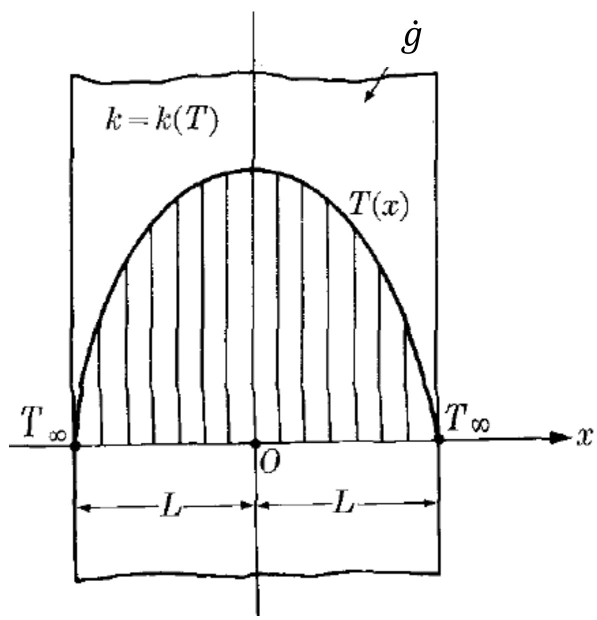


Figure 4.6: Details for Example 7.

Solution:

The formulation of the problem is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \dot{g} = 0 \qquad (4\text{-}56)$$

$$\frac{dT(0)}{dx} = 0 \qquad \text{and} \qquad T(L) = T_\infty$$

Employing the one-dimensional form of Eq. (4-53), $\frac{d\theta}{dt} = \frac{k}{k_R}\frac{dT}{dt}$

We may transform Eq. (4-56) to

$$\frac{d^2\theta}{dx^2} + \frac{\dot{g}}{k_R} = 0 \qquad \frac{d\theta(0)}{dx} = 0 \qquad \text{and} \qquad \theta(L) = \theta_\infty$$

where, according to Eq. (4-52), $\theta = \frac{1}{k_R}\int_{T_R}^{T_\infty} k(T)\, dT$

The solution of Eq. (4-56) is $\frac{\theta(x) - \theta_\infty}{\dot{g}L^2/2k_R} = 1 - \left(\frac{x}{L}\right)^2$ (4-57)

Introducing Eqs. (θ) and (θ_∞) into Eq. (4-57), we obtain the temperature of the plate in terms of T as follows:

$\frac{\frac{1}{k_R}\int_{T_\infty}^{T} k(T)dT}{\dot{g}L^2/2k_R} = 1 - \left(\frac{x}{L}\right)^2$ For the special case $k = k_R$ (l +(βT), the equation becomes

$$\frac{[T(x) - T_\infty] + \left(\frac{\beta}{2}\right)[T^2(x) - T^2{}_\infty]}{\dot{g}L^2/2k_R} = 1 - \left(\frac{x}{L}\right)^2$$

University of Anbar
College of Engineering
Mechanical Engineering Dept.

Advanced Heat Transfer/ I
Conduction and Radiation

Handout Lectures for MSc. / Power
Chapter Five
Transient Conduction Heat Transfer

Course Tutor

Assist. Prof. Dr. Waleed M. Abed

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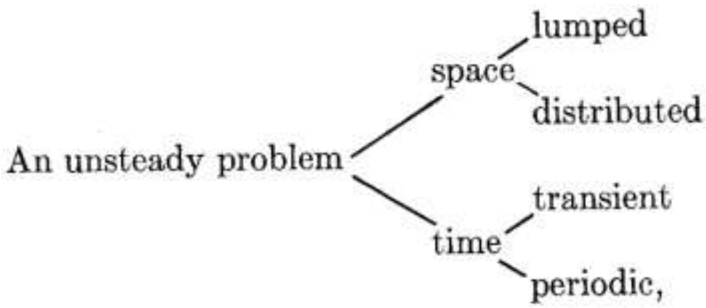
Chapter Five
Transient Conduction Heat Transfer

5.1 Introduction

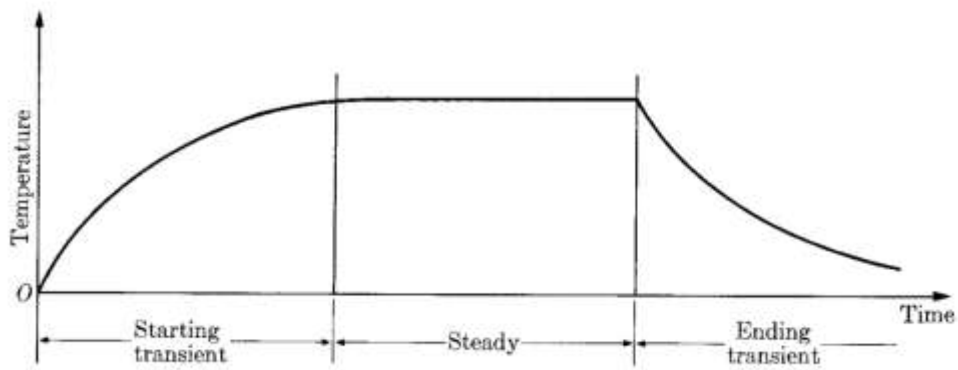
Many heat transfer problems are *time dependent*. Such *unsteady*, or *transient*, problems typically arise when the boundary conditions of a system are changed. For example, if the surface temperature of a system is altered, the temperature at each point in the system will also begin to change. The changes will continue to occur until a steady-state temperature distribution is reached. Consider a hot metal billet that is removed from a furnace and exposed to a cool airstream. Energy is transferred by convection and radiation from its surface to the surroundings. Energy transfer by conduction also occurs from the interior of the metal to the surface, and the temperature at each point in the billet decreases until a steady-state condition is reached. The final properties of the metal will depend significantly on the time-temperature history that results from heat transfer. Controlling the heat transfer is one key to fabricating new materials with enhanced properties.

Our objective in this chapter is to develop procedures for determining the time dependence of the temperature distribution within a solid during a transient process, as well as for determining heat transfer between the solid and its surroundings. The nature of the procedure depends on assumptions that may be made for the process. If, for example, temperature gradients within the solid may be neglected, a comparatively simple approach, termed the *lumped capacitance* method, may be used to determine the variation of temperature with time.

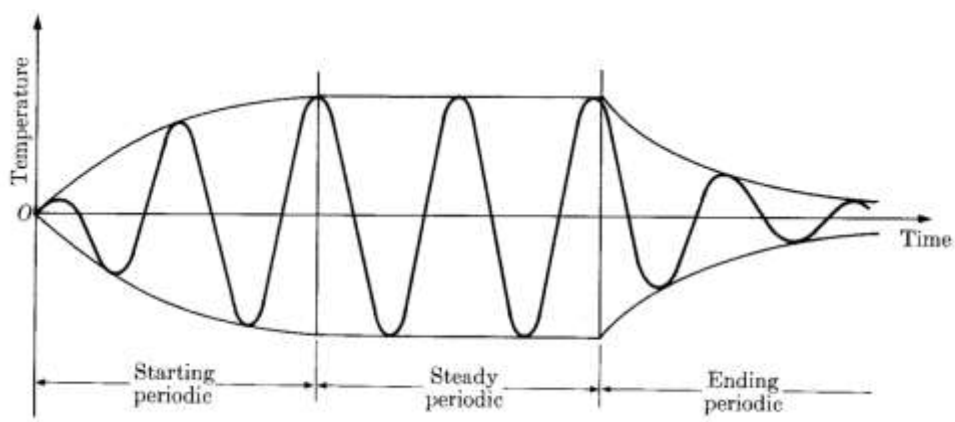
Transient problems can be classified with respect to their dependence on space (as lumped or distributed), and since then formulated them accordingly. Also, these problems may be classified with respect to their dependence on time (as transient or periodic). Thus,



Where, each transient or periodic problem involves a *starting*, a *steady*, and an *ending* time interval as shown in Figure 5.1.



Unsteady Transient



Unsteady Periodic

Figure 5.1: The behavior of unsteady transient and unsteady periodic.

5.2 The Lumped Capacitance Method

A simple transient conduction problem is one for which a solid experiences a sudden change in its thermal environment. Consider a hot metal forging that is initially at a uniform temperature T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$ (see Figure 5.2). If the quenching is said to begin at time $t = 0$, the temperature of the solid will decrease for time $t > 0$, until it eventually reaches T_∞. This reduction is due to convection heat transfer at the solid–liquid interface. The essence of the *lumped capacitance method* is the assumption that "*the temperature of the solid is spatially uniform at any instant during the transient process*". This assumption implies that temperature gradients within the solid are negligible.

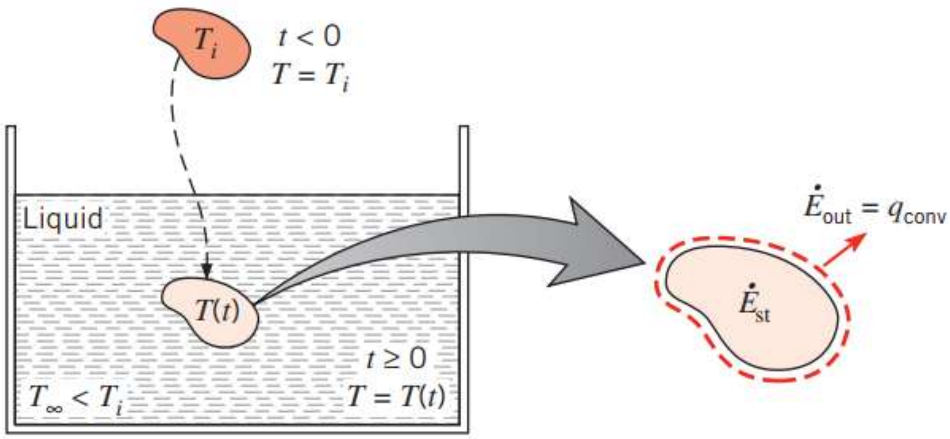


Figure 5.2: Cooling of a hot metal forging.

From *Fourier's law*, heat conduction in the absence of a temperature gradient implies the existence of infinite thermal conductivity. Such a condition is clearly impossible. However, the condition is closely approximated if the resistance to conduction within the solid is small compared with the resistance to heat transfer between the solid and its surroundings. For now we assume that this is, in fact, the case.

In neglecting temperature gradients within the solid, we can no longer consider the problem from within the framework of the heat equation, since the heat equation is a differential equation governing the spatial temperature distribution within the solid. Instead, the transient temperature response is determined by formulating an overall *energy balance* on the entire solid. This balance must relate the rate of heat loss at the surface to the rate of change of the internal energy.

$$-E_{out} = E_{st} \tag{5-1}$$

Or, $$-hA_s(T - T_\infty) = \rho VC \frac{dT}{dt} \tag{5-2}$$

Introducing the temperature difference

$$\theta = T - T_\infty \tag{5-3}$$

and recognizing that $(d\theta/dt) = (dT/dt)$ if T_∞ is constant, it follows that

$$\frac{\rho VC}{hA_s}\frac{d\theta}{dt} = -\theta \tag{5-4}$$

Separating variables and integrating from the initial condition, for which $t = 0$ and $T(0) = T_i$, we then obtain

$$\frac{\rho VC}{hA_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt \tag{5-5}$$

Where, $\theta_i = T_i - T_\infty$, thus, evaluating the integrals, it follows that

$$\frac{\rho VC}{hA_s}\ln\frac{\theta_i}{\theta} = t \tag{5-6}$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = e^{\left[-\left(\frac{hA_s}{\rho VC}\right)t\right]} \tag{5-7}$$

Equation (5.6) may be used to determine the *time* required for the solid to reach some temperature T, or, conversely, Equation 5.7 may be used to compute the *temperature* reached by the solid at some time t.

The foregoing results indicate that the difference between the solid and fluid temperatures must decay exponentially to *zero* as t approaches infinity. This behavior is shown in Figure 5.3. From Equation 5.7 it is also evident that the quantity $(\rho VC/hA_s)$ may be interpreted as a thermal time constant expressed as

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho VC) = R_t C_t \tag{5-8}$$

where, R_t is the resistance to convection heat transfer and C_t is the lumped thermal capacitance of the solid. Any increase in R_t or C_t will cause a solid to respond more slowly to changes in its thermal environment. This behavior is analogous to the voltage decay that occurs when a capacitor is discharged through a resistor in an electrical *RC* circuit.

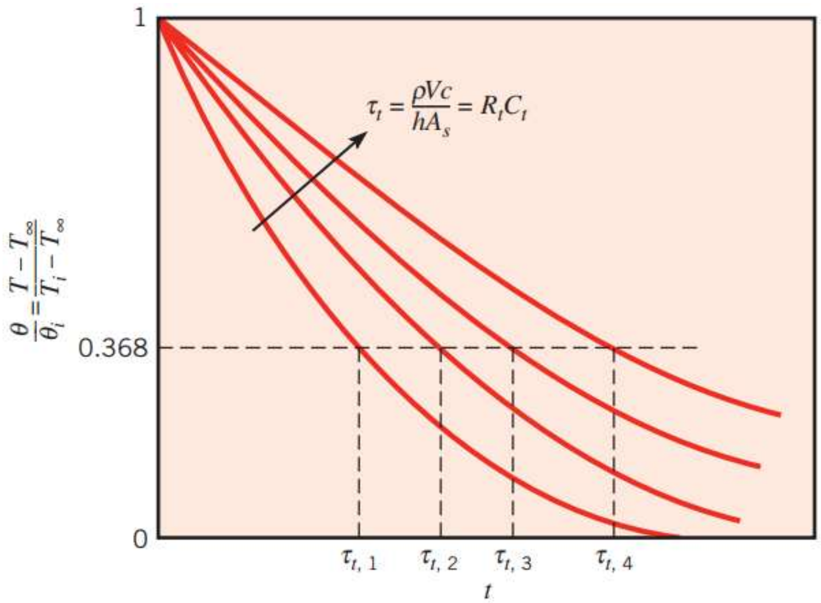


Figure 5.3: Transient temperature response of lumped capacitance solids for different thermal time constants *t*.

To determine the total energy transfer Q occurring up to sometime t, we simply write

$$Q = \int_0^t q\, dt = hA_s \int_0^t \theta\, dt \tag{5-9}$$

Substituting for θ from Equation 5.7 and integrating, we obtain

$$Q = (\rho VC)\theta_i \left[1 - e^{\left[-\left(\frac{hA_s}{\rho VC}\right)t\right]}\right] = (\rho VC)\theta_i \left[1 - e^{\left[\frac{-t}{\tau_t}\right]}\right] \tag{5-10}$$

The quantity Q is, of course, related to the change in the internal energy of the solid.

$$-Q = \Delta E_{st} \tag{5-11}$$

For quenching, Q is positive and the solid experiences a decrease in energy. Equations 5.6, 5.7, and 5.10 also apply to situations where the solid is heated ($\theta < 0$), in which case Q is negative and the internal energy of the solid increases.

5.3 Validity of the Lumped Capacitance Method

From the foregoing results it is easy to see why there is a strong preference for using the lumped capacitance method. It is certainly the simplest and most convenient method that can be used to solve transient heating and cooling problems. Hence it is important to determine under what conditions it may be used with reasonable accuracy.

To develop a suitable criterion considers steady-state conduction through the plane wall of area A (Figure 5.4). Although we are assuming steady-state conditions, the following criterion is readily extended to transient processes. One surface is maintained at a temperature $T_{s,1}$ and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$. The temperature of this surface will be some intermediate value $T_{s,2}$, for which $T_\infty < T_{s,2} < T_{s,1}$. Hence under steady-state conditions the surface energy balance reduces to

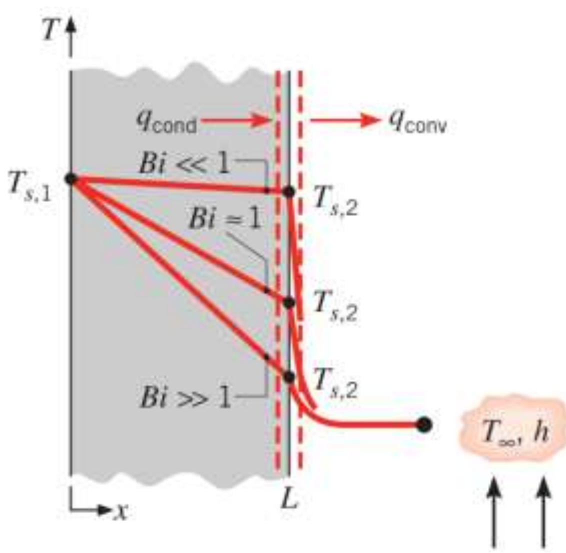


Figure 5.4: Effect of *Biot number* on steady-state temperature distribution in a plane wall with surface convection.

$$\frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) = hA_s\left(T_{s,2} - T_\infty\right) \qquad (5\text{-}12)$$

where k is the thermal conductivity of the solid. Rearranging, we then obtain

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,cond.}}{R_{t,conv.}} = \frac{hL}{k} = Bi \qquad (5\text{-}13)$$

The quantity (hL/k) appearing in Equation 5.13 is a dimensionless parameter. It is termed the *Biot number*, and it plays a fundamental role in conduction problems that involve surface convection effects. According to Equation 5.13 and as illustrated in Figure 5.4, the *Biot number* provides a measure of the temperature drop in the solid relative to the temperature difference between the solid's surface and the fluid. From Equation 5.13, it is also evident that the *Biot number* may be interpreted as a ratio of thermal resistances. In particular, if $Bi \ll 1$, *the resistance to conduction within the solid is much less than the resistance to convection across the fluid boundary layer*. Hence, *the assumption of a uniform temperature distribution within the solid is reasonable if the Biot number is small*.

Although we have discussed the *Biot number* in the context of steady-state conditions, we are reconsidering this parameter because of its significance to transient conduction problems. Consider the plane wall of Figure 5.5, which is initially at a uniform temperature T_i and experiences convection cooling when it is immersed in a fluid of $T_\infty < T_i$. The problem may be treated as one-dimensional in x, and we are interested in the temperature variation with position and time, $T(x, t)$. This variation is a strong function of the *Biot number*, and three conditions are shown in Figure 5.5. Again, for $Bi \ll 1$ the temperature gradients in the solid are small and the assumption of a uniform temperature distribution, $T(x, t) \approx T(t)$ is reasonable. Virtually all the temperature difference is between the solid and the fluid, and the solid temperature remains nearly uniform as it decreases to T_∞. For moderate to large values of the *Biot number*, however, the temperature gradients within the solid are significant. Hence $T = T(x, t)$. Note that for $Bi \gg 1$, the temperature difference across the solid is much larger than that between the surface and the fluid.

We conclude this section by emphasizing the importance of the lumped capacitance method. Its inherent simplicity renders it the preferred method for solving transient heating and cooling problems. Hence, *when confronted with such a problem, the very first thing that one should do is calculate the Biot number*. If the following condition is satisfied

$$\frac{hL_c}{k} = Bi < 0.1 \qquad (5\text{-}14)$$

The error associated with using the lumped capacitance method is small. For convenience, it is customary to define the *characteristic length* of Equation 5.14 as the ratio of the solid's

volume to surface area $L_c = V/A_s$. Such a definition facilitates calculation of L_c for solids of complicated shape and reduces to the half-thickness L for a plane wall of thickness $2L$ (Figure 5.5), to $r_o/2$ for a long cylinder, and to $r_o/3$ for a sphere. However, if one wishes to implement the criterion in a conservative fashion, L_c should be associated with the length scale corresponding to the maximum spatial temperature difference. Accordingly, for a symmetrically heated (or cooled) plane wall of thickness $2L$, L_c would remain equal to the half-thickness L. However, for a long cylinder or sphere, L_c would equal the actual radius r_o, rather than $r_o/2$ or $r_o/3$.

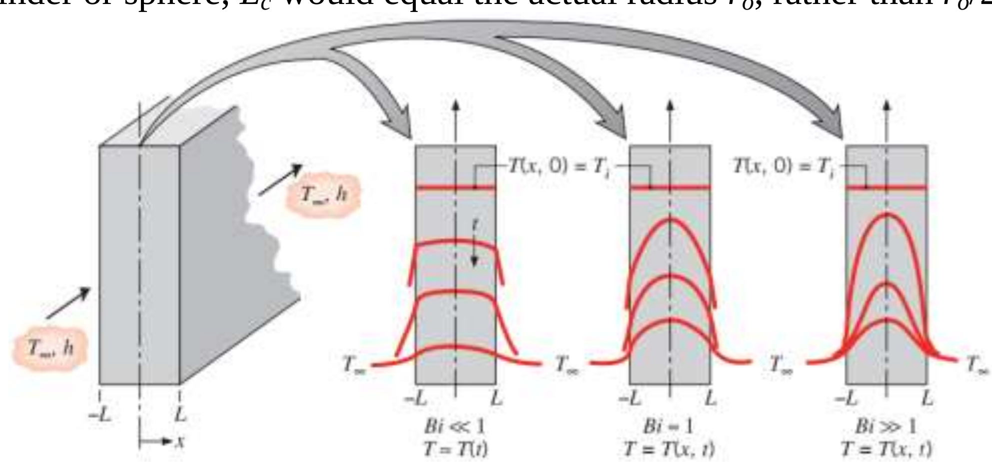


Figure 5.5: Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

Finally, we note that, with $L_c = V/A_s$, the exponent of Equation 5.7 may be expressed as

$$\frac{hA_s\,t}{\rho VC} = \frac{ht}{\rho C L_c} = \frac{hL_c}{k}\frac{k}{\rho C}\frac{t}{{L_c}^2} = \frac{hL_c}{k}\frac{\alpha t}{{L_c}^2} = Bi \times Fo \qquad (5\text{-}15)$$

Example: A steel ball bearing (diameter = 25 mm, ρ_{Steel} = 7833 kg/m^3 and C_{Steel} = 0.465 kJ/kg.°C) is heated in a furnace to a temperature of 750 °C. It is then removed to be quenched in water at 25 °C. If the time required for transferring the ball bearing between the furnace and the water is 8 sec through the atmospheric air environment (air temperature 20 °C). **Determine** by using lumped system analysis, the time it takes the ball bearing to cool to 200 °C. Taking, convection heat transfer coefficients in the air and water are 30 W/m^2.°C and 3000 W/m^2.°C, respectively.

Solution:

$$\frac{T_{(t)} - T_\infty}{T_i - T_\infty} = e^{\frac{-hA_S}{\rho VC} \times t}$$

$$\frac{V}{A} = L_c = \frac{r_o}{3} = \frac{25\times10^{-3}}{2\times3} = 4.2 \times 10^{-3}\ (m)$$

$$\frac{T_{(8)} - 20}{750 - 20} = e^{\left[\frac{-30\times8}{7833\times0.465\times10^3\times4.2\times10^{-3}}\right]} = 0.98443$$

$$T_{(8)} = 738.6\ °C$$

$$\frac{200-25}{738.6-25} = e^{\left[\frac{-3000\times t}{7833\times0.465\times10^3\times4.2\times10^{-3}}\right]}$$

$$-1.4055 = -0.19611 \times t$$

$$t = 7.167\ (sec.)$$

5.4 Distributed Systems Having Stepwise Disturbances

5.4.1 Cartesian Geometry:

For distributed systems having stepwise disturbances, no mathematic beyond that introduced in Chapter 4 is needed. Difficulties arising from nonhomogeneous boundary conditions or differential equations may be eliminated, as with steady problems, by introducing a *change in temperature level* or by *superposition*. Some remarks no non-homogeneities may, however, be helpful. In this connection, let us consider the following four problems.

Therefore, with no *internal generation* and the assumption of *constant thermal conductivity*, the energy equation can then be reduced to,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5-16}$$

To solve Equation 5.16 for the temperature distribution *T*(*x*, *t*), it is necessary to specify an *initial condition* and *two boundary conditions*. For the typical transient conduction problem of Figure 5.5, the initial condition is

$$T_{(x,0)} = T_i$$

and the boundary conditions are

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

and,

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T_{(L,t)} - T_\infty\right]$$

($T_{(x,0)} = T_i$) presumes a uniform temperature distribution at time $t = 0$; ($\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$) reflects the symmetry requirement for the midplane of the wall; and ($-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T_{(L,t)} - T_\infty\right]$) describes the surface condition experienced for time $t > 0$. From Equations 5.16 through initial and boundary conditions above, it is evident that, in addition to depending on x and t, temperatures in the wall also depend on a number of physical parameters. In particular

$T = T(x, t, T_i, T_\infty, L, k, \alpha, h)$

Example 1: A plate of thickness $2L$ having the uniform initial temperature T_o is plunged suddenly into a bath at the constant temperature T_∞ as shown in Figure 5.6. The heat transfer coefficient is large. Find the unsteady temperature of the plate.

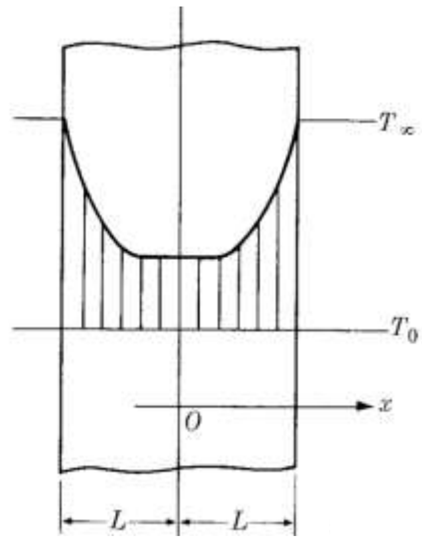


Figure 5.6

Solution:

In terms of ($\theta = T - T_\infty$) and the x-axis of Figure 5.6, the formulation of the problem is,

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta_{(x,0)} = \theta_i = T_i - T_\infty$$

$$\left.\frac{\partial\theta_{(0,t)}}{\partial x}\right|_{x=0} = 0$$

$$\theta_{(L,t)} = 0$$

Assume the existence of a product solution of the form (Eq. 5.16)

$$\theta_{(x,t)} = X_{(x)}\,\tau_{(t)}$$

Note that only the x-axis yields a characteristic-value problem; then, with the proper choice of separation constant, the product solution ($\theta_{(x,t)} = X_{(x)}\,\tau_{(t)}$) gives,

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0 \qquad \rightarrow \qquad \left.\frac{\partial X_{(0)}}{\partial x}\right|_{x=0} = 0 \text{ and } \quad X_{(L)} = 0$$

$$X = C_1\cos\lambda x + C_2\sin\lambda x$$

$$X' = -\lambda\, C_1\sin\lambda x + \lambda\, C_2\cos\lambda x$$

From B.C. $[\left.\frac{\partial X_{(0)}}{\partial x}\right|_{x=0} = 0] \rightarrow 0 = -\lambda\, C_1\sin(\lambda\times 0) + \lambda\, C_2\cos(\lambda\times 0) \rightarrow C_2 = 0$

From B.C. $[X_{(L)} = 0] \rightarrow 0 = C_1 \cos \lambda L \rightarrow 0 = \cos \lambda L \rightarrow$

$\lambda_n L = (2n+1) \times \frac{\pi}{2}$ $n = 0, 1, 2, 3, \ldots$ (Characteristic values)

$\lambda_n = (2n+1) \times \frac{\pi}{2L}$

$X = A_n \cos \lambda_n x$

$\frac{d\tau}{dt} + \alpha\lambda^2 \tau = 0 \rightarrow \frac{d\tau}{\tau} = -\alpha\lambda^2 dt \rightarrow \tau(t) = C_3 e^{-\alpha\lambda^2 t} \rightarrow \tau_n(t) = B_n e^{-\alpha\lambda_n^2 t}$

Therefore, the product solution becomes, $\theta_{(x,t)} = \sum_{n=0}^{\infty} A_n \cos \lambda_n x \; B_n e^{-\alpha\lambda_n^2 t}$

Let $C_n = A_n \times B_n \rightarrow \theta_{(x,t)} = \sum_{n=0}^{\infty} C_n \cos \lambda_n x \; e^{-\alpha\lambda_n^2 t}$

Finally, introducing the initial condition $[\theta_{(x,0)} = \theta_i]$ into transient temperature distribution ($\theta_{(x,t)}$) gives

$\theta_i = \sum_{n=0}^{\infty} C_n \cos \lambda_n x \; e^{-\alpha\lambda_n^2 (0)} \rightarrow \theta_i = \sum_{n=0}^{\infty} C_n \cos \lambda_n x$

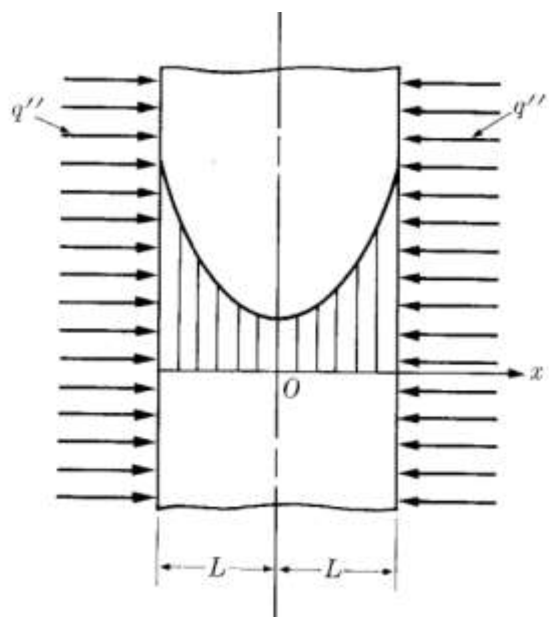
The above equation is the *Fourier cosine series* expansion of θ_i over the interval (0, L). The coefficient (C_n) may be evaluated in the usual manner. The result is

$C_n = (-1)^n \frac{2\theta_i}{\lambda_n L}$

Thus, the unsteady temperature of the plate to be as,

$\frac{\theta_{(x,t)}}{\theta_i} = \frac{T_{(x,t)} - T_\infty}{T_i - T_\infty} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n L} e^{-\alpha\lambda_n^2 t} \cos \lambda_n x$

Example 2: The constant heat flux (q'') is applied to both surfaces of a flat plate of thickness 2L as shown in Figure 5.7.The initial temperature of the plate is T_∞. Find the unsteady temperature of the plate.



Solution:

In terms of ($\theta = T - T_\infty$) and the x-axis of Figure 5.7, the formulation of the problem is,

$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t}$

$\theta_{(x,0)} = 0$

Figure 5.7

$\left.\frac{\partial \theta_{(0,t)}}{\partial x}\right|_{x=0} = 0$

$k\frac{\partial \theta_{(L,t)}}{\partial x} = q''$

Let us try to solve the problem by the assumption of below equation as,

$\theta_{(x,t)} = \Psi_{(x,t)} + \Phi_{(x)}$

The separation constant that is selected to make the *x*-axis of $\Psi_{(x,t)}$ a *characteristic-value* problem forces $\Psi_{(x,t)}$ to be an exponentially decaying function in time, hence in the limit $\Psi_{(x,t)} \to 0$ and $\theta_{(x,t)} \to \Phi_{(x)}$ as $t \to \infty$. This result violates the physics of the problem, the temperature of the plate should increase without limit as $t \to \infty$. To satisfy this condition, we modify the above equation by the addition of the term $\varphi_{(t)}$ such that $\varphi_{(t)} \to \infty$ as $t \to \infty$. Thus the proper assumption is,

$\theta_{(x,t)} = \Psi_{(x,t)} + \Phi_{(x)} + \varphi_{(t)}$

Now in terms of above equation, the formulation of the problem becomes

$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \Psi}{\partial t}, \quad \Psi_{(x,0)} = -\Phi_{(x)} - \varphi_{(0)}, \quad \left.\frac{\partial \Psi_{(0,t)}}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial \Psi_{(L,t)}}{\partial x}\right|_{x=L} = 0$

$\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \varphi}{\partial t}, \quad \left.\frac{d\Phi_{(0)}}{dx}\right|_{x=0} = 0, \quad +k\frac{d\Phi_{(L)}}{dx} = q''$

Since $\varphi_{(t)}$ and $\Phi_{(x)}$ can vary independently, ($\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \varphi}{\partial t}$) holds when it is equal to a constant, say *C*. Then the general solution of ($\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \varphi}{\partial t}$) is obtained in the form

$\varphi_{(t)} = \alpha C t + C_1$

$\Phi_{(x)} = \frac{1}{2}Cx^2 + C_2 x + C_3$

Here C_2 and C may readily be evaluated by introducing ($\Phi_{(x)}$) equation into ($+k\frac{d\Phi_{(L)}}{dx} = q''$) equation. The result is C_2= 0, $C = q''/kL$. Hence, $\varphi_{(t)} =$

$q"t/\rho CL + C_1$ and $\Phi_{(x)} = q"x^2/2kL + C_3$, where C_1 and C_3 are the remaining constants. However, noting that the solution of $\Psi_{(x,t)}$ depends on $\varphi_{(t)}$ and $\Phi_{(x)}$, may arbitrarily set these constants equal to *zero*. Thus,

$\varphi_{(t)} = \frac{q"t}{\rho CL}$ and $\Phi_{(x)} = \frac{q"x^2}{2kL}$

On the other hand, the product solution $\Psi_{(x,t)} = X_{(x)}\tau_{(t)}$ applied to $(\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \Psi}{\partial t})$ results in

$\frac{d^2X}{dx^2} + \lambda^2 X = 0$, $\frac{dX_{(0)}}{dx} = 0$, $\frac{dX_{(L)}}{dx} = 0$

The solution of above equation is,

$X_n(x) = A_n \cos\lambda_n x$ Characteristic functions

$\lambda_n = \frac{n\pi}{L}$ where, n= 0, 1, 2, 3, ….. Characteristic values

$\frac{d\tau}{dt} + \alpha\lambda^2\tau = 0$

The solution of above equation is,

$\tau_n = C_n e^{-\alpha\lambda_n^2 t}$

Thus the product solution of $\Psi_{(x,t)}$ yields

$\Psi_{(x,t)} = a_o + \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t}\cos\lambda_n x$

Where, $a_o = A_o C_o$ and $a_n = A_n C_n$

Finally, the initial value of above equation, which is equal to $-\Phi_{(x)}$, gives

$-\frac{q"x^2}{2kL} = a_o + \sum_{n=1}^{\infty} a_n \cos\lambda_n x$

The coefficients a_o and a_n are,

$a_o = -\frac{q"L}{6k}$ and $a_n = -(-1)^n \frac{2q"L}{k(\lambda_n L)^2}$

Therefore, the unsteady temperature disturibution of the plate is,

$$\frac{\theta_{(x,t)}}{q"L/k} = \frac{\alpha t}{L^2} + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(\lambda_n L)^2} e^{-\alpha\lambda_n^2 t}\cos\lambda_n x$$

5.4.2 Cylindrical Geometry

Example 3: An infinitely long rod of radius R having the uniform initial temperature T_o is plunged suddenly into a bath at temperature T_∞ as shown in Figure 5.8. The heat transfer coefficient is large. Find the expression of unsteady temperature disturibution of the rod.

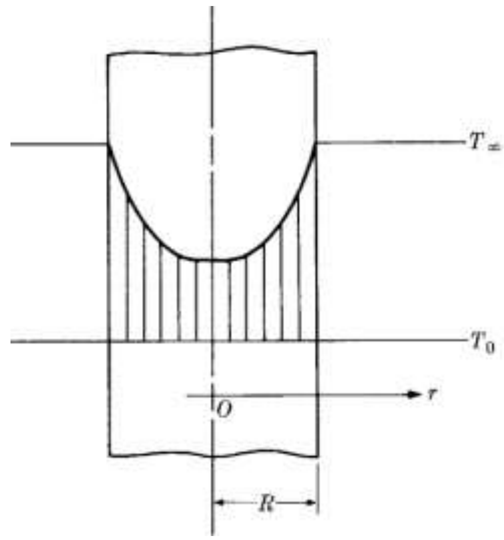


Figure 5.8

Solution:

In terms of ($\theta = T - T_\infty$) and the r-axis of Figure 5.8, the formulation of the problem is,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$$\theta_{(r,0)} = \theta_o = T_o - T_\infty$$

$$\theta_{(R,t)} = 0$$

$$\left.\frac{\partial \theta_{(0,t)}}{\partial r}\right|_{r=0} = 0$$

The product solution $\theta_{(r,t)} = \mathcal{R}_{(r)}\tau_{(t)}$ introduced into formulation equation results in

$$\frac{d}{dr}\left(r\,\frac{d\mathcal{R}}{dr}\right) + \lambda^2 r\mathcal{R} = 0 \;; \qquad \mathcal{R}_{(R)} = 0 \;, \qquad \left.\frac{d\mathcal{R}_{(0)}}{dr}\right|_{r=0} = 0$$

The solution of above equation is,

$$\mathcal{R}_{n_{(r)}} = A_n J_o(\lambda_n r) \qquad \text{Characteristic functions}$$

And the zeros of ($J_o(\lambda_n R) = 0$) are the characteristic values. The solution of equation ($\frac{d\tau}{dt} + \alpha\lambda^2\tau = 0$) is,

$$\tau_{n_{(t)}} = C_n e^{-\alpha\lambda_n^2 t}$$

Hence the product solution becomes

$$\theta_{(r,t)} = \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t} J_o(\lambda_n r)$$

Where , $a_n = A_n C_n$

The initial value of above equation is,

$$\theta_o = \sum_{n=1}^{\infty} a_n J_o(\lambda_n r)$$

The above equation is the *Fourier-Bessel series* expansion of θ_o. Here the coefficient a_n may be calculated in the usual manner. The result is,

$$a_n = \frac{2\theta_o}{(\lambda_n R)\ J_1(\lambda_n R)}$$

Thus, introducing the a_n equation into $\theta_{(r,t)}$ equation, the unsteady temperature disturibution of the rod is,

$$\theta_{(r,t)} = \frac{T_{(r,t)} - T_\infty}{T_o - T_\infty} = 2\sum_{n=1}^{\infty} \frac{e^{-\alpha\lambda_n^2 t}\ J_o(\lambda_n r)}{(\lambda_n R)\ J_1(\lambda_n R)}$$

5.4.3 Spherical Geometry

Example 4: A solid sphere of radius *R* having the uniform initial temperature T_o is plunged suddenly into a bath at temperature T_∞ as shown in Figure 5.9. The heat transfer coefficie temperature disturibution of th

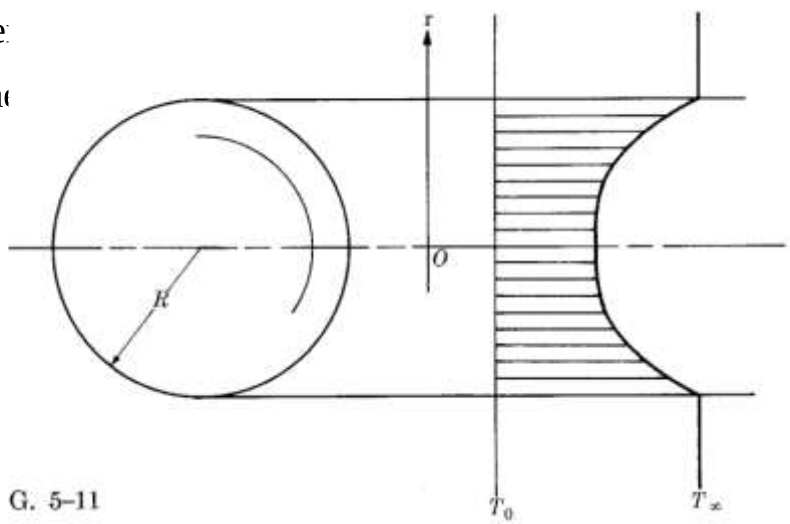


Figure 5.9

Solution:

The formulation of the problem in terms of ($\theta = T - T_\infty$) is,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

$$\theta_{(r,0)} = \theta_o = T_o - T_\infty$$

$$\theta_{(R,t)} = 0$$

$$\left.\frac{\partial\theta_{(0,t)}}{\partial r}\right|_{r=0} = 0 \quad \text{or} \quad \theta_{(0,t)} = finite$$

The product solution $\theta_{(r,t)} = \mathcal{R}_{(r)}\tau_{(t)}$ introduced into formulation equation yields

$$\frac{d}{dr}\left(r^2\frac{d\mathcal{R}}{dr}\right) + \lambda^2 r^2\mathcal{R} = 0 \;; \qquad \mathcal{R}_{(R)} = 0 \;, \qquad \left.\frac{d\mathcal{R}_{(0)}}{dr}\right|_{r=0} = 0$$

Particular solutions corresponding to the differential equation of above equation are,

$$J_{1/2}(\lambda r)/r^{1/2} \qquad \text{and} \qquad J_{-1/2}(\lambda r)/r^{1/2}$$

Furthermore, noting from *Bessel series* properties that

$$J_{1/2}(\lambda r) \sim \sin\lambda r/r^{1/2} \qquad \text{and} \qquad J_{-1/2}(\lambda r) \sim \cos\lambda r/r^{1/2}$$

The solutions of above equations may be rearranged to give,

$$\sin\lambda r/r \qquad \text{and} \qquad \cos\lambda r/r$$

This result explains the use of the well-known transformation

$$\theta_{(r,t)} = \Psi_{(r,t)}/r$$

$$\frac{\partial^2\Psi}{\partial r^2} = \frac{1}{\alpha}\frac{\partial\Psi}{\partial t}$$

$$\Psi_{(r,0)} = r\theta_o = r(T_o - T_\infty)$$

$$\Psi_{(R,t)} = 0$$

$$\Psi_{(0,t)} = 0$$

Hence, the problem is reduced to a problem of Cartesian geometry.

The product solution $\Psi_{(r,t)} = \mathcal{R}_{(r)}\tau_{(t)}$ applied to ($\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\Psi}{\partial t}$) yields

$\frac{d^2\mathcal{R}}{dr^2} + \lambda^2\mathcal{R} = 0$, $\mathcal{R}_{(0)} = 0$ and $\mathcal{R}_{(R)} = 0$

The solution of above equation is,

$\mathcal{R}_n(r) = A_n \sin \lambda_n r$ Characteristic functions

$\lambda_n = \frac{n\pi}{R}$ where, n= 0, 1, 2, 3, ….. Characteristic values

The solution of equation ($\frac{d\tau}{dt} + \alpha\lambda^2\tau = 0$) is,

$\tau_{n(t)} = C_n e^{-\alpha\lambda_n^2 t}$

Hence the product solution becomes

$\Psi_{(r,t)} = \sum_{n=1}^{\infty} a_n e^{-\alpha\lambda_n^2 t} \sin(\lambda_n r)$

Where , $a_n = A_n C_n$

$r\theta_o = \sum_{n=1}^{\infty} a_n \sin(\lambda_n r)$

The coefficient a_n is

$a_n = (-1)^{n+1} \frac{2\theta_o}{\lambda_n}$

Finally, the unsteady temperature disturibution of the sphere is found to be,

$\theta_{(r,t)} = \frac{T_{(r,t)} - T_\infty}{T_o - T_\infty} = 2\sum_{n=1}^{\infty}(-1)^{n+1} e^{-\alpha\lambda_n^2 t} \frac{\sin \lambda_n r}{\lambda_n r}$

5.5 The Numerical Method of Transient Heat Conduction

Analytical solutions to transient problems are restricted to simple geometries and boundary conditions, such as the one-dimensional cases considered in the preceding sections. For some simple *two-* and *three-dimensional* geometries, analytical solutions are still possible. However, in many cases the geometry and/or boundary conditions preclude the use of analytical techniques, and recourse must be made to *finite-difference* (or *finite-element*) methods.

5.1.1 Discretization of the Heat Equation:

- ✓ **The Explicit Method**

Once again consider the two-dimensional system of Figure 5.10. Under transient conditions with constant properties and no internal generation, the appropriate form of the heat equation, Equation 1-18-b, is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (5\text{-}17)$$

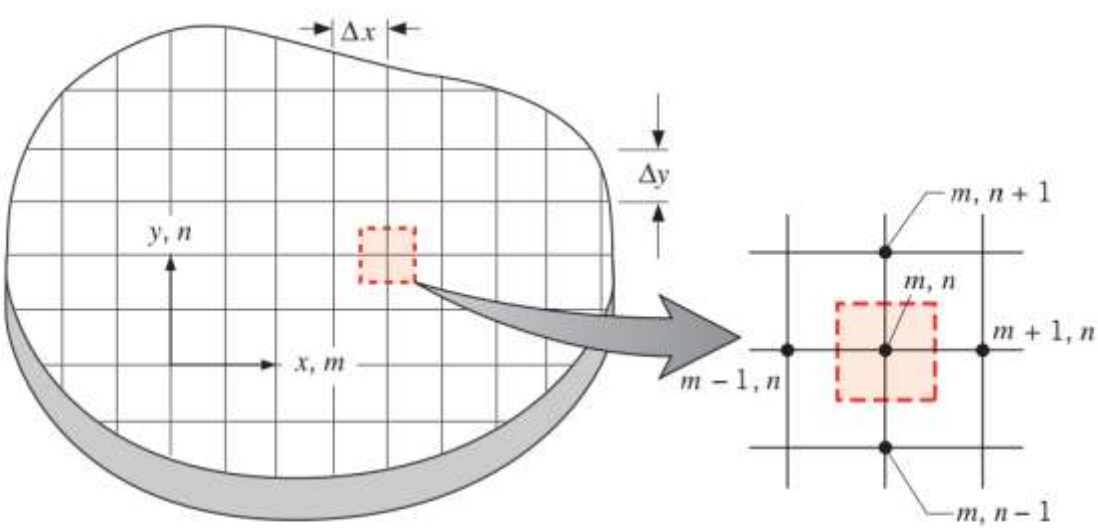


Figure 5.10: Two-dimensional conduction, Nodal network.

To obtain the *finite-difference* form of this equation, we may use the *central-difference* approximations to the spatial derivatives prescribed as,

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} = \frac{T_{m+1,n}-T_{m,n}}{\Delta x} \qquad \left.\frac{\partial T}{\partial x}\right|_{m-1/2,n} = \frac{T_{m,n}-T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} = \frac{\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right|_{m-1/2,n}}{\Delta x}$$

Therefore,

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} = \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2} \qquad (5\text{-}18)$$

Proceeding in a similar fashion, it is readily shown that

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} = \frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2} \qquad (5\text{-}19)$$

Once again the *m* and *n* subscripts may be used to designate the *x*- and *y*-locations of discrete nodal points. However, in addition to being discretized in space, the problem must be discretized in time. The integer *p* is introduced for this purpose, where, $t = p\,\Delta t$

and the *finite-difference* approximation to the time derivative in Equation 5.17 is expressed as

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} = \frac{T^{p+1}{}_{m,n}-T^{p}{}_{m,n}}{\Delta t} \qquad (5\text{-}20)$$

The superscript *p* is used to denote the time dependence of *T*, and the time derivative is expressed in terms of the difference in temperatures associated with the *new* (*p*+1) and *previous* (or *old*) (*p*) times. Hence calculations must be performed at successive times separated by the interval Δt, and just as a *finite-difference* solution restricts temperature determination to discrete points in space, it also restricts it to discrete points in time.

If Equations 5.18, 5.19 and 5.20 is substituted into Equation 5.17, the nature of the *finite-difference* solution will depend on the specific time at which temperatures are evaluated in the *finite-difference* approximations to the spatial derivatives. In the explicit method of solution, these temperatures are evaluated at the *previous* (*p*) time. Hence Equation 5.20 is considered to be a *forward-difference* approximation to the time derivative. Evaluating terms on the *right-hand* side of Equations 5.17 at *p* and substituting into Equation 5.17, the explicit form of the *finite-difference* equation for the interior node (*m*, *n*) is

$$\frac{T^{P}{}_{m+1,n}+T^{P}{}_{m-1,n}-2T^{P}{}_{m,n}}{(\Delta x)^2}+\frac{T^{P}{}_{m,n+1}+T^{P}{}_{m,n-1}-2T^{P}{}_{m,n}}{(\Delta y)^2}=\frac{1}{\alpha}\,\frac{T^{p+1}{}_{m,n}-T^{p}{}_{m,n}}{\Delta t} \qquad (5\text{-}21)$$

Solving for the nodal temperature at the *new* (*p*+1) time and assuming that $\Delta x = \Delta y$, it follows that

$$T^{p+1}{}_{m,n}=Fo\left(T^{P}{}_{m+1,n}+T^{P}{}_{m-1,n}+T^{P}{}_{m,n+1}+T^{P}{}_{m,n-1}\right)+(1-4Fo)T^{p}{}_{m,n}$$

(5-22)

where *Fo* is a *finite-difference* form of the *Fourier number*, which is the ratio of diffusive or conductive transport rate to the quantity storage rate.

$$Fo=\frac{\alpha\,\Delta t}{(\Delta x)^2} \qquad (5\text{-}23)$$

This approach can easily be extended to *one*- or *three-dimensional* systems. If the system is *one-dimensional* in *x*, the explicit form of the *finite-difference* equation for an interior node *m* reduces to

$$T^{p+1}{}_{m}=Fo\,(T^{P}{}_{m+1}+T^{P}{}_{m-1})+(1-2Fo)T^{p}{}_{m} \qquad (5\text{-}24)$$

Equations 5.22 and 5.24 are *explicit* because *unknown* nodal temperatures for the new time are determined exclusively by *known* nodal temperatures at the previous time. Hence calculation of the unknown temperatures is straightforward. Since the temperature of each interior node is known at *t*= 0 (*p*= 0) from prescribed initial conditions, the calculations begin at *t*= Δt (*p*= 1), where Equation 5.22 or 5.24 is applied to each interior node to determine its temperature. With

With temperatures known for *t*= Δt, the appropriate finite-difference equation is then applied at each node to determine its temperature at *t*= 2Δt (*p*= 2). In this way, the transient temperature distribution is obtained by *marching out in time*, using intervals of Δt.

The accuracy of the *finite-difference* solution may be improved by decreasing the values of Δx and Δt. Of course, the number of interior nodal points that must be considered increases with decreasing Δx, and the number of time intervals required to carry the solution to a prescribed final time increases with decreasing Δt. Hence the computation time increases with decreasing Δx and Δt. The choice of Δx is typically based on a compromise between accuracy and computational requirements. Once this selection has been made, however, the value of Δt may not be chosen independently. It is, instead, determined by stability requirements.

An undesirable feature of the *explicit* method is that it is not unconditionally *stable*. In a transient problem, the solution for the nodal temperatures should continuously approach final (steady-state) values with increasing time. However, with the explicit method, this solution may be characterized by numerically induced oscillations, which are physically impossible. The oscillations may become *unstable,* causing the solution to diverge from the actual steady-state conditions. To prevent such erroneous results, the prescribed value of Δt must be maintained below a certain limit, which depends on Δx and other parameters of the system. This dependence is termed a stability criterion, which may be obtained mathematically or demonstrated from a thermodynamic argument. For the problems of interest in this text, *the criterion is determined by requiring that the coefficient associated with the node of interest at the previous time is greater than or equal to zero*.

In general, this is done by collecting all terms involving ${T^p}_{m,n}$ to obtain the form of the coefficient. This result is then used to obtain a limiting relation involving *Fo*, from which the maximum allowable value of Δt may be determined. For example, with Equations 5.22 and 5.24 already expressed in the desired form, it follows that the stability criterion for a *one-dimensional* interior node is (1 - 2*Fo*) ≥ 0, or $Fo \leq \frac{1}{2}$ and for a *two-dimensional* node, it is (1 - 4*Fo*) ≥ 0, or $o \leq \frac{1}{4}$.

For prescribed values of Δx and α, these criteria may be used to determine upper limits to the value of Δt.

Equations 5.22 and 5.24 may also be derived by applying the *energy balance* method to a control volume about the interior node. Accounting for changes in thermal energy storage, a general form of the *energy balance* equation may be expressed as

$$E_{in} + E_g = E_{st} \tag{5-25}$$

In the interest of adopting a consistent methodology, it is again assumed that all heat flow is *into* the node.

Example 5: Consider the surface node of the *one-dimensional* system shown in Figure 5.11. To more accurately determine thermal conditions near the surface, this node has been assigned a thickness that is one-half that of the interior nodes. Assuming convection transfer from an adjoining fluid and no generation, it follows from Equation 5.25 that

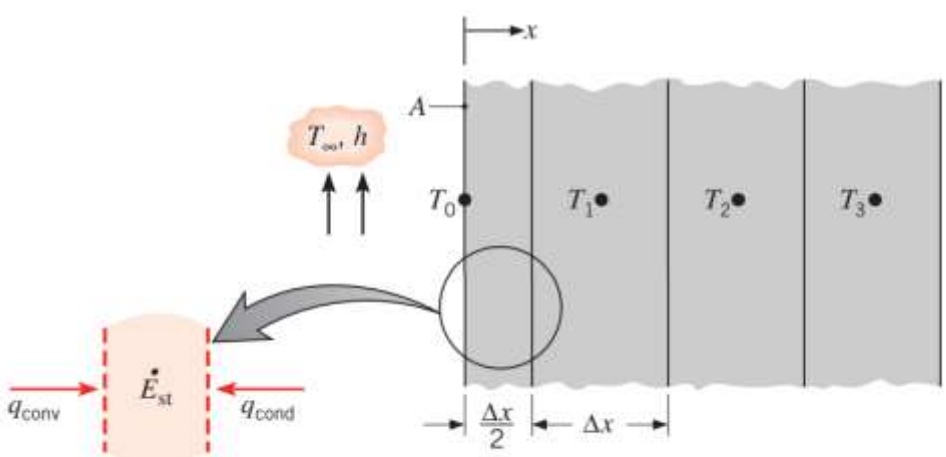


Figure 5.11: Surface node with convection and one-dimensional transient conduction.

$$hA\left(T_\infty - T_o^p\right) + \frac{kA}{\Delta x}\left(T_1^p - T_o^p\right) = \rho CA\frac{\Delta x}{2}\frac{\left(T_o^{p+1} - T_o^p\right)}{\Delta t} \quad (5\text{-}26)$$

or, solving for the surface temperature at $t + \Delta t$,

$$T_o^{p+1} = \frac{2h\Delta t}{\rho C\Delta x}\left(T_\infty - T_o^p\right) + \frac{2\alpha\Delta x}{\Delta x^2}\left(T_1^p - T_o^p\right) + T_o^p \quad (5\text{-}27)$$

Recognizing that $(2h\Delta t/\rho C\Delta x) = 2(h\Delta x/k)(\alpha\Delta t/\Delta x^2) = 2BiFo$ and grouping terms involving T_o^p, it follows that

$$T_o^{p+1} = 2Fo\left(T_1^p - BiT_\infty\right) + (1 - 2Fo - 2BiFo)T_o^p \quad (5\text{-}28)$$

The *finite-difference* form of the *Biot number* is

$$Bi = \frac{h\Delta x}{k}$$

Recalling the procedure for determining the stability criterion, we require that the coefficient for be greater than or equal to zero. Hence

$$1 - 2Fo - 2BiFo \geq 0 \quad (5\text{-}29)$$

Or, $Fo(1 + Bi) \leq \frac{1}{2}$

Since the complete finite-difference solution requires the use of Equation 5.24 for the interior nodes, as well as Equation 5.28 for the surface node, Equation 5.29 must be contrasted with ($Fo \leq \frac{1}{2}$) to determine which requirement is more stringent. Since $Bi \geq 0$, it is apparent that the limiting value of Fo for Equation 5.29 is less than that for Equation 5.82. To ensure stability for all nodes, Equation 5.29 should therefore be used to select the maximum allowable value of Fo, and hence Δt, to be used in the calculations.

Example 6: A fuel element of a nuclear reactor is in the shape of a plane wall of thickness $2L= 20$ mm as shown in Figure 5.12 and is convectively cooled at both surfaces, with $h= 1100$ W/m^{2}K and $T_\infty= 250$ °C. At normal operating power, heat is generated uniformly within the element at a volumetric rate of $q_1= 10^7$ W/m^3. A departure from the steady-state conditions associated with normal operation will occur if there is a change in the generation rate. Consider a sudden change to $q_2= 2\times10^7$ W/m^3, and use the *explicit finite-difference* method to determine the fuel element temperature distribution after 1.5 s. The fuel element thermal properties are $k= 30$ W/mK and $\alpha= 5\times10^{-6}$ m^2/s.

Solution:

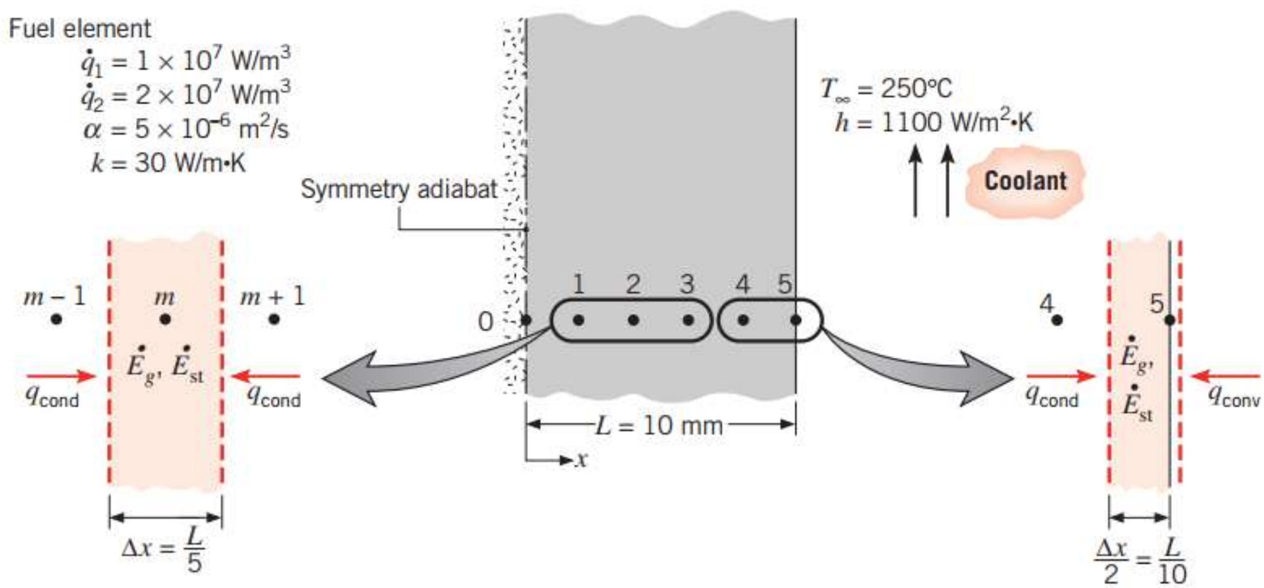


Figure 5.12: A rectangular fuel element with surface cooling.

A numerical solution will be obtained using a space increment of $\Delta x= 2$ mm. Since there is symmetry about the midplane, the nodal network yields six unknown nodal temperatures. Using the *energy balance* method, Equation 5.25, an *explicit finite-difference* equation may be derived for any interior node *m*.

$$kA\,\frac{T_{m-1}^{p}-T_{m}^{p}}{\Delta x}+kA\,\frac{T_{m+1}^{p}-T_{m}^{p}}{\Delta x}+\ qA\,\Delta x=\rho A\,\Delta x\,c\,\frac{T_{m}^{p+1}-T_{m}^{p}}{\Delta t}$$

Solving for T_m^{p+1} and rearranging,

$$T_{m}^{p+1}=Fo\left[T_{m-1}^{p}+T_{m+1}^{p}+\frac{q(\Delta x)^{2}}{k}\right]+(1-2Fo)T_{m}^{p} \qquad (1)$$

This equation may be used for node 0, with $T_{m-1}^{p} = T_{m+1}^{p}$, as well as for nodes 1, 2, 3, and 4. Applying energy conservation to a control volume about node 5,

$$hA(T_{\infty}-T_{5}^{p})+kA\,\frac{T_{4}^{p}-T_{5}^{p}}{\Delta x}+qA\,\frac{\Delta x}{2}=\rho A\,\frac{\Delta x}{2}\,c\,\frac{T_{5}^{p+1}-T_{5}^{p}}{\Delta t}$$

$$T_{5}^{p+1}=2Fo\left[T_{4}^{p}+Bi\,T_{\infty}+\frac{q(\Delta x)^{2}}{2k}\right]+(1-2Fo-2Bi\,Fo)T_{5}^{p} \qquad (2)$$

Since the most restrictive stability criterion is associated with Equation 2, we select Fo from the requirement that

$$Fo(1+Bi)\le\frac{1}{2}$$

$$Bi=\frac{h\,\Delta x}{k}=\frac{1100\ \mathrm{W/m^{2}\cdot K}\ (0.002\ \mathrm{m})}{30\ \mathrm{W/m\cdot K}}=0.0733 \qquad Fo\le 0.466$$

$$\Delta t=\frac{Fo(\Delta x)^{2}}{\alpha}\le\frac{0.466(2\times10^{-3}\ \mathrm{m})^{2}}{5\times10^{-6}\ \mathrm{m^{2}/s}}\le 0.373\ \mathrm{s}$$

To be well within the stability limit, we select Δt= 0.3 s, which corresponds to

$$Fo=\frac{5\times10^{-6}\ \mathrm{m^{2}/s}(0.3\ \mathrm{s})}{(2\times10^{-3}\ \mathrm{m})^{2}}=0.375$$

Substituting numerical values, including q_2= 2×10^{7} W/m^3, the nodal equations become

$$T_0^{p+1} = 0.375(2T_1^p + 2.67) + 0.250T_0^p$$

$$T_1^{p+1} = 0.375(T_0^p + T_2^p + 2.67) + 0.250T_1^p$$

$$T_2^{p+1} = 0.375(T_1^p + T_3^p + 2.67) + 0.250T_2^p$$

$$T_3^{p+1} = 0.375(T_2^p + T_4^p + 2.67) + 0.250T_3^p$$

$$T_4^{p+1} = 0.375(T_3^p + T_5^p + 2.67) + 0.250T_4^p$$

$$T_5^{p+1} = 0.750(T_4^p + 19.67) + 0.195T_5^p$$

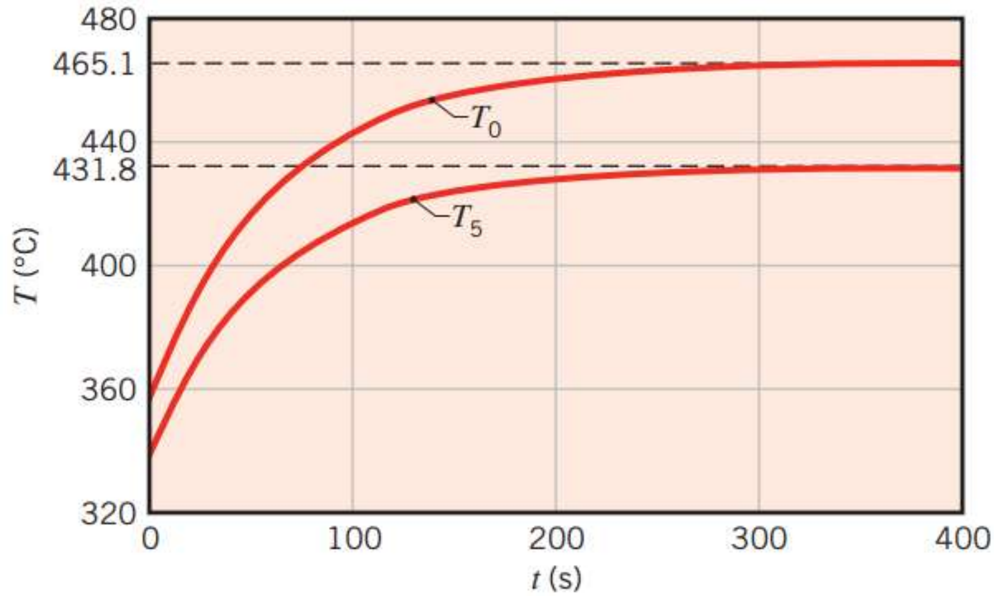
$$T_5 = T_\infty + \frac{qL}{h} = 250°\text{C} + \frac{10^7\ \text{W/m}^3 \times 0.01\ \text{m}}{1100\ \text{W/m}^2\cdot\text{K}} = 340.91°\text{C}$$

$$T(x) = 16.67\left(1 - \frac{x^2}{L^2}\right) + 340.91°\text{C}$$

Computed temperatures for the nodal points of interest are shown in the first row of the accompanying table. Using the finite-difference equations, the nodal temperatures may be sequentially calculated with a time increment of 0.3 s until the desired final time is reached. The results are illustrated in rows 2 through 6 of the table and may be contrasted with the new steady-state condition (row 7)

Tabulated Nodal Temperatures

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5
0	0	357.58	356.91	354.91	351.58	346.91	340.91
1	0.3	358.08	357.41	355.41	352.08	347.41	341.41
2	0.6	358.58	357.91	355.91	352.58	347.91	341.88
3	0.9	359.08	358.41	356.41	353.08	348.41	342.35
4	1.2	359.58	358.91	356.91	353.58	348.89	342.82
5	1.5	360.08	359.41	357.41	354.07	349.37	343.27
∞	∞	465.15	463.82	459.82	453.15	443.82	431.82



✓ The Implicit Method

In the *explicit finite-difference* scheme, the temperature of any node at $t+\Delta t$ may be calculated from knowledge of temperatures at the same and neighboring nodes for the *preceding time t*. Hence determination of a nodal temperature at some time is *independent* of temperatures at other nodes for the *same time*. Although the method offers computational convenience, it suffers from limitations on the selection of Δt. For a given space increment, the time interval must be compatible with stability requirements. Frequently, this dictates the use of extremely small values of Δt, and a very large number of time intervals may be necessary to obtain a solution.

A reduction in the amount of computation time may often be realized by employing an *implicit*, rather than *explicit*, *finite-difference* scheme. The *implicit* form of a finite-difference equation may be derived by using Equation 5.20 to approximate the time derivative, while evaluating all other temperatures at the *new* (p+1) time, instead of the *previous* (p) time. Equation 5.20 is then considered to provide a *backward-difference* approximation to the time derivative. In contrast to

Equation 5.21, the *implicit* form of the *finite-difference* equation for the interior node of a *two-dimensional* system is then

$$\frac{T^{P+1}{}_{m+1,n}+T^{P+1}{}_{m-1,n}-2T^{P+1}{}_{m,n}}{(\Delta x)^2}+\frac{T^{P+1}{}_{m,n+1}+T^{P+1}{}_{m,n-1}-2T^{P+1}{}_{m,n}}{(\Delta y)^2}=\frac{1}{\alpha}\,\frac{T^{p+1}{}_{m,n}-T^{p}{}_{m,n}}{\Delta t}$$

(5-30)

Rearranging and assuming $\Delta x = \Delta y$, it follows that

$$T^{p}{}_{m,n}=(1+4Fo)T^{p+1}_{m,n}-Fo\left(T^{p+1}_{m+1,n}+T^{p+1}_{m-1,n}+T^{p+1}_{m,n+1}+T^{p+1}_{m,n-1}\right) \qquad (5\text{-}31)$$

From Equation 5.31 it is evident that the new temperature of the (m, n) node depends on the *new* temperatures of its adjoining nodes, which are, in general, unknown. Hence, to determine the unknown nodal temperatures at $t+\Delta t$, the corresponding nodal equations must be *solved simultaneously*. Such a solution may be affected by using *Gauss–Seidel* iteration or matrix inversion. The *marching* solution would then involve *simultaneously solving* the nodal equations at each time $t=\Delta t$, $t=2\Delta t$, ….., until the desired final time was reached.

Relative to the *explicit* method, the *implicit* formulation has the important advantage of being *unconditionally stable*. That is, the solution remains stable for all space and time intervals, in which case there are no restrictions on Δx and Δt. Since larger values of Δt may therefore be used with an implicit method, computation times may often be reduced, with little loss of accuracy. Nevertheless, to maximize accuracy, Δt should be sufficiently small to ensure that the results are independent of further reductions in its value.

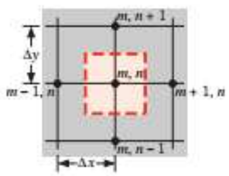
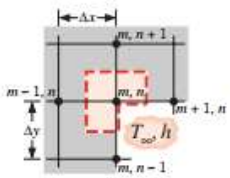
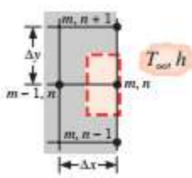
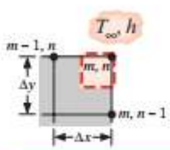
The *implicit* form of a *finite-difference* equation may also be derived from the *energy balance* method. For the surface node of Figure 5.11, it is readily shown that

$$T^{p}_{o}+2FoBiT_{\infty}=(1+2Fo+2FoBi)T^{p+1}_{o}-2FoT^{p+1}_{1} \qquad (5\text{-}32)$$

For any interior node of Figure 5.11, it may also be shown that

$$(1+2Fo)T^{p+1}_{m}-Fo\left(T^{p+1}_{m-1}+T^{p+1}_{m+1}\right)=T^{p}_{m} \qquad (5\text{-}33)$$

Forms of the implicit finite-difference equation for other common geometries are presented in Table below. Each equation may be derived by applying the *energy balance* method.

Conguration	(*a*) Explicit Method: Finite-Difference Equation	(*a*) Explicit Method: Stability Criterion	(*b*) Implicit Method
	$T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}) + (1-4Fo)T_{m,n}^{p}$ (5.79) 1. Interior node	$Fo \leq \frac{1}{4}$ (5.83)	$(1+4Fo)T_{m,n}^{p+1} - Fo(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p}$ (5.95)
	$T_{m,n}^{p+1} = \frac{2}{3}Fo(T_{m+1,n}^{p} + 2T_{m-1,n}^{p} + 2T_{m,n+1}^{p} + T_{m,n-1}^{p} + 2Bi\,T_{\infty}) + (1 - 4Fo - \frac{4}{3}Bi\,Fo)T_{m,n}^{p}$ (5.88) 2. Node at interior corner with convection	$Fo(3+Bi) \leq \frac{3}{4}$ (5.89)	$(1+4Fo(1+\frac{1}{3}Bi))T_{m,n}^{p+1} - \frac{2}{3}Fo \cdot (T_{m+1,n}^{p+1} + 2T_{m-1,n}^{p+1} + 2T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + \frac{4}{3}Bi\,Fo\,T_{\infty}$ (5.98)
	$T_{m,n}^{p+1} = Fo(2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p} + 2Bi\,T_{\infty}) + (1 - 4Fo - 2Bi\,Fo)T_{m,n}^{p}$ (5.90) 3. Node at plane surface with convection[a]	$Fo(2+Bi) \leq \frac{1}{2}$ (5.91)	$(1+2Fo(2+Bi))T_{m,n}^{p+1} - Fo(2T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + 2Bi\,Fo\,T_{\infty}$ (5.99)
	$T_{m,n}^{p+1} = 2Fo(T_{m-1,n}^{p} + T_{m,n-1}^{p} + 2Bi\,T_{\infty}) + (1 - 4Fo - 4Bi\,Fo)T_{m,n}^{p}$ (5.92) 4. Node at exterior corner with convection	$Fo(1+Bi) \leq \frac{1}{4}$ (5.93)	$(1+4Fo(1+Bi))T_{m,n}^{p+1} - 2Fo(T_{m-1,n}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + 4Bi\,Fo\,T_{\infty}$ (5.100)

[a] To obtain the finite-difference equation and/or stability criterion for an adiabatic surface (or surface of symmetry), simply set *Bi* equal to zero.

Example 7: A thick slab of copper initially at a uniform temperature of 20°C is suddenly exposed to radiation at one surface such that the net heat flux is maintained at a constant value of 3×10^5 W/m^2 (see Figure 5.13). Using the explicit and implicit finite-difference techniques with a space increment of Δx = 75 mm, determine the temperature at the irradiated surface and at an interior point that is 150 mm from the surface after 2 min have elapsed. Compare the results with those obtained from an appropriate analytical solution.

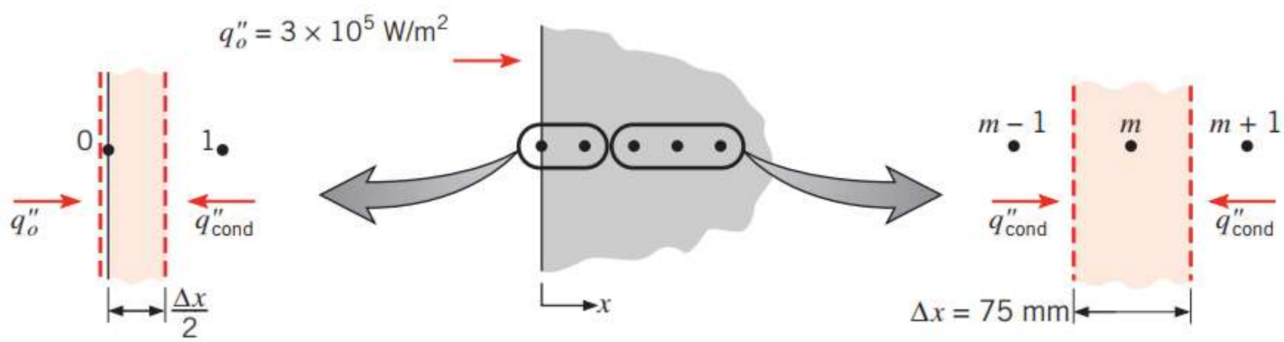


Figure 5.13: A Thick slab of copper, initially at a uniform temperature.

Solution:

Properties: Copper (300 K): k= 401 W/mK, α= 117×10^{-6} m^2/s.

An *explicit* form of the *finite-difference* equation for the surface node may be obtained by applying an *energy balance* to a control volume about the node.

$$q''_o A + kA\frac{T_1^p - T_0^p}{\Delta x} = \rho A\frac{\Delta x}{2}c\frac{T_0^{p+1} - T_0^p}{\Delta t}$$

$$T_0^{p+1} = 2Fo\left(\frac{q''_0 \Delta x}{k} + T_1^p\right) + (1 - 2Fo)T_0^p$$

The *finite-difference* equation for any interior node is given by Equation 5.29. Both the surface and interior nodes are governed by the *stability criterion*

$$Fo \le \frac{1}{2}$$

Noting that the *finite-difference* equations are simplified by choosing the maximum allowable value of *Fo*, we select *Fo*= 0.5. Hence

$$\Delta t = Fo\frac{(\Delta x)^2}{\alpha} = \frac{1}{2}\,\frac{(0.075\text{ m})^2}{117 \times 10^{-6}\text{ m}^2/\text{s}} = 24\text{ s}$$

$$\frac{q''_o\,\Delta x}{k} = \frac{3 \times 10^5\text{ W/m}^2\,(0.075\text{ m})}{401\text{ W/m}\cdot\text{K}} = 56.1°\text{C}$$

the *finite-difference* equations become

$$T_0^{p+1} = 56.1°\text{C} + T_1^p \qquad \text{and} \qquad T_m^{p+1} = \frac{T_{m+1}^p + T_{m-1}^p}{2}$$

for the surface and interior nodes, respectively. Performing the calculations, the results are tabulated as follows:

Explicit Finite-Difference Solution for $Fo = \frac{1}{2}$

p	t(s)	T_0	T_1	T_2	T_3	T_4
0	0	20	20	20	20	20
1	24	76.1	20	20	20	20
2	48	76.1	48.1	20	20	20
3	72	104.2	48.1	34.0	20	20
4	96	104.2	69.1	34.0	27.0	20
5	120	125.2	69.1	48.1	27.0	23.5

After 2 min, the surface temperature and the desired interior temperature are T_o 125.2°C and T_2= 48.1°C.

To determine the extent to which the accuracy may be improved by reducing *Fo*, let us redo the calculations for *Fo* = 1/4 (Δt = 12 s). The *finite-difference* equations are then of the form

$$T_0^{p+1} = \frac{1}{2}(56.1°\mathrm{C} + T_1^p) + \frac{1}{2}T_0^p$$

$$T_m^{p+1} = \frac{1}{4}(T_{m+1}^p + T_{m-1}^p) + \frac{1}{2}T_m^p$$

and the results of the calculations are tabulated as follows:

Explicit Finite-Difference Solution for $Fo = \frac{1}{4}$

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20	20	20	20	20	20	20	20	20
1	12	48.1	20	20	20	20	20	20	20	20
2	24	62.1	27.0	20	20	20	20	20	20	20
3	36	72.6	34.0	21.8	20	20	20	20	20	20
4	48	81.4	40.6	24.4	20.4	20	20	20	20	20
5	60	89.0	46.7	27.5	21.3	20.1	20	20	20	20
6	72	95.9	52.5	30.7	22.5	20.4	20.0	20	20	20
7	84	102.3	57.9	34.1	24.1	20.8	20.1	20.0	20	20
8	96	108.1	63.1	37.6	25.8	21.5	20.3	20.0	20.0	20
9	108	113.6	67.9	41.0	27.6	22.2	20.5	20.1	20.0	20.0
10	120	118.8	72.6	44.4	29.6	23.2	20.8	20.2	20.0	20.0

After 2 min, the desired temperatures are T_o= 118.8°C and T_2= 44.4°C. Comparing the above results with those obtained for , it is clear that by reducing *Fo* we have diminished the problem of recurring temperatures. We have also predicted greater thermal penetration (to node 6 instead of node 3). An assessment of the improvement in accuracy will be given later, by comparison with an exact solution. In the absence of an exact solution, the value of *Fo* could be successively reduced until the results became essentially independent of *Fo*.

Performing an *energy balance* on a control volume about the surface node, the implicit form of the finite-difference equation is

$$q''_o + k\,\frac{T_1^{p+1} - T_0^{p+1}}{\Delta x} = \rho\,\frac{\Delta x}{2}\,c\,\frac{T_0^{p+1} - T_0^{p}}{\Delta t}$$

$$(1 + 2Fo)T_0^{p+1} - 2FoT_1^{p+1} = \frac{2\alpha q''_o\,\Delta t}{k\,\Delta x} + T_0^{p}$$

Arbitrarily choosing $Fo = \frac{1}{2}$ ($\Delta t = 24$ s), it follows that

$$2T_0^{p+1} - T_1^{p+1} = 56.1 + T_0^{p}$$

$$-T_{m-1}^{p+1} + 4T_m^{p+1} - T_{m+1}^{p+1} = 2T_m^{p}$$

In contrast to the *explicit* method, the *implicit* method requires the simultaneous solution of the nodal equations for all nodes at time p+1. Hence, the number of nodes under consideration must be limited to some finite number, and a boundary condition must be applied at the last node. The number of nodes may be limited to those that are affected significantly by the change in boundary condition for the time of interest. From the results of the explicit method, it is evident that we are safe in choosing nine nodes corresponding to T_o, T_1, . . . , T_8. We are thereby assuming that, at t= 120 s, there has been no change in T_9, and the boundary condition is implemented numerically as T_9= 20°C.

We now have a set of nine equations that must be solved simultaneously for each time increment. We can express the equations in the form [A][T]= [C],

$$[A] = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 56.1 + T_0^p \\ 2T_1^p \\ 2T_2^p \\ 2T_3^p \\ 2T_4^p \\ 2T_5^p \\ 2T_6^p \\ 2T_7^p \\ 2T_8^p + T_9^{p+1} \end{bmatrix}$$

Note that numerical values for the components of [*C*] are determined from previous values of the nodal temperatures. Note also how the finite-difference equation for node 8 appears in matrices [*A*] and [*C*], with $T^{p+1}_9 = 20°C$, as indicated previously. A table of nodal temperatures may be compiled, beginning with the first row ($p = 0$) corresponding to the prescribed initial condition. To obtain nodal temperatures for subsequent times, the matrix equation must be solved. At each time step $p+1$, [*C*] is updated using the previous time step (p) values. The process is carried out five times to determine the nodal temperatures at 120 s. The desired temperatures are T_o= 114.7°C and T_2= 44.2°C.

Implicit Finite-Difference Solution for $Fo = \frac{1}{2}$

p	t(s)	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0	20.0
1	24	52.4	28.7	22.3	20.6	20.2	20.0	20.0	20.0	20.0
2	48	74.0	39.5	26.6	22.1	20.7	20.2	20.1	20.0	20.0
3	72	90.2	50.3	32.0	24.4	21.6	20.6	20.2	20.1	20.0
4	96	103.4	60.5	38.0	27.4	22.9	21.1	20.4	20.2	20.1
5	120	114.7	70.0	44.2	30.9	24.7	21.9	20.8	20.3	20.1

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Advanced Heat Transfer/ I
Conduction and Radiation

Handout Lectures for MSc. / Power
Chapter Six
Radiation Heat Transfer

Course Tutor

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Chapter Six
Radiation Heat Transfer

6.1 Introduction

Heat transfer by conduction and convection requires the presence of a temperature gradient in some form of matter. In contrast, heat transfer by thermal radiation requires no matter. It is an extremely important process, and in the physical sense it is perhaps the most interesting of the heat transfer modes. It is relevant to many industrial heating, cooling, and drying processes, as well as to energy conversion methods that involve fossil fuel combustion and solar radiation.

6.2 Processes and Properties of Radiation

6.2.1 Fundamental Concepts

Consider a solid that is initially at a higher temperature T_s than that of its surroundings T_{sur}, but around which there exists a vacuum (see Figure 6.1). The presence of the vacuum precludes energy loss from the surface of the solid by conduction or convection. However, our intuition tells us that the solid will cool and eventually achieve thermal equilibrium with its surroundings. This cooling is associated with a reduction in the internal energy stored by the solid and is a direct consequence of the *emission* of thermal radiation from the surface. In turn, the surface will intercept and absorb radiation originating from the surroundings. However, if $T_s > T_{sur}$ the net heat transfer rate by radiation $q_{rad,net}$ is from the surface, and the surface will cool until T_s reaches T_{sur}.

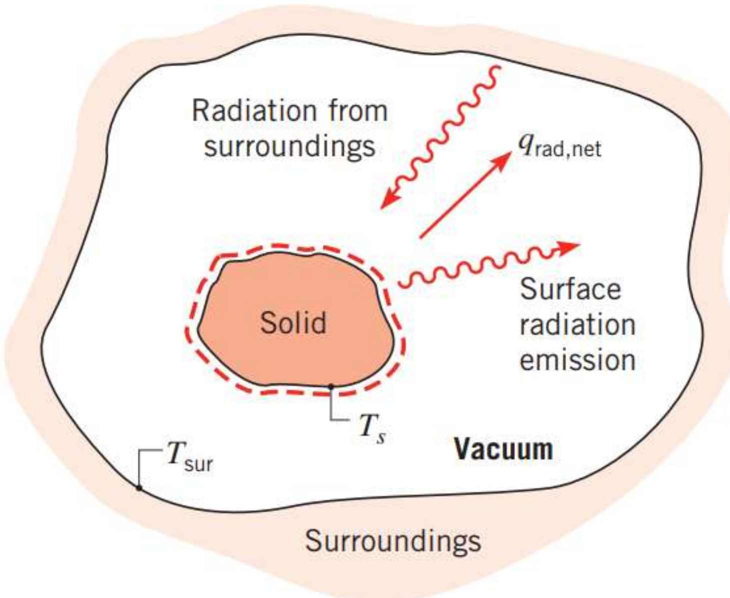


Figure 6.1: Radiation cooling of a hot solid.

We associate thermal radiation with the rate at which energy is emitted by matter as a result of its finite temperature. At this moment thermal radiation is being emitted by all the matter that surrounds you: by the furniture and walls of the room, if you are indoors, or by the ground, buildings, and the atmosphere and sun if you are outdoors. The mechanism of emission is related to energy released as a result of oscillations or transitions of the many electrons that constitute matter. These oscillations are, in turn, sustained by the internal energy, and therefore the temperature, of the matter. Hence we associate the emission of thermal radiation with thermally excited conditions within the matter.

We know that radiation originates due to *emission* by matter and that its subsequent transport does not require the presence of any matter. But *what is the nature of this transport*? One theory views radiation as the propagation of a collection of particles termed *photons* or *quanta*. Alternatively, radiation may be viewed as the propagation of *electromagnetic waves*. In any case we wish to attribute to radiation the standard wave properties of *frequency* ν and *wavelength* λ. For radiation propagating in a particular medium, the two properties are related by

$$\lambda = \frac{c}{\nu} \quad (6\text{-}1)$$

where c is the speed of light in the medium. For propagation in a vacuum, c_o = 2.998×10^8 m/s. The unit of wavelength is commonly the micrometer (μm), where 1 μm 10^{-6} m.

The complete electromagnetic spectrum is delineated in Figure 6.2. The short wavelength *gamma* rays, *X* rays, and ultraviolet (*UV*) radiation are primarily of interest to the high-energy physicist and the nuclear engineer, while the long wavelength microwaves and radio waves ($\lambda > 10^5$ μm) are of concern to the electrical engineer. It is the intermediate portion of the spectrum, which extends from approximately 0.1 to 100 μm and includes a portion of the *UV* and all of the visible and infrared (*IR*), that is termed *thermal radiation* because it is both caused by and affects the thermal state or temperature of matter. For this reason, *thermal radiation* is pertinent to heat transfer.

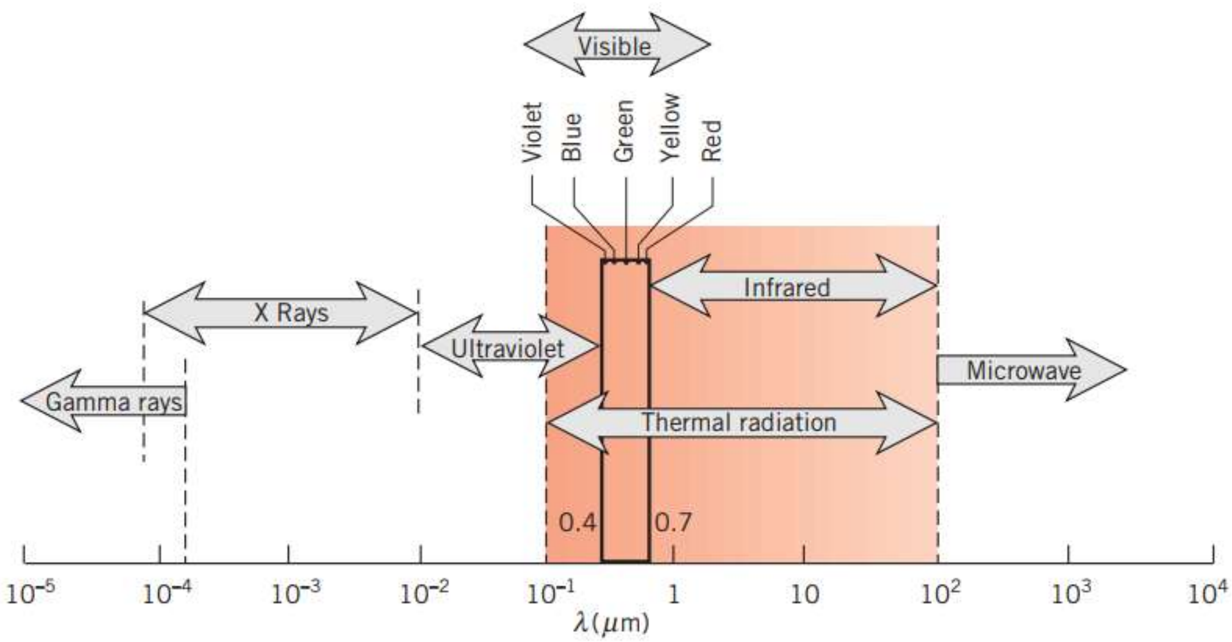


Figure 6.2: Spectrum of electromagnetic radiation.

6.2.2 Radiation Heat Fluxes

Various types of heat fluxes are pertinent to the analysis of radiation heat transfer. Table 6.1 lists four distinct radiation fluxes that can be defined at a surface. The *emissive power*, E (W/m^2), is the rate at which radiation is emitted from a surface per unit surface area, over all wavelengths and in all directions. This *emissive power* was related to the behavior of a *blackbody* through the relation $E = \varepsilon\sigma T_s^4$, where ε is a surface property known as the *emissivity*.

Table 6.1: Radiative fluxes (over all wavelengths and in all directions).

Flux (W/m^2)	Description	Comment
Emissive power, E	Rate at which radiation is emitted from a surface per unit area	$E = \varepsilon\sigma T_s^4$
Irradiation, G	Rate at which radiation is incident upon a surface per unit area	Irradiation can be reflected, absorbed, or transmitted
Radiosity, J	Rate at which radiation leaves a surface per unit area	For an opaque surface $J = E + \rho G$
Net radiative flux, $q''_{rad} = J - G$	Net rate of radiation leaving a surface per unit area	For an opaque surface $q''_{rad} = \varepsilon\sigma T_s^4 - \alpha G$

Radiation from the surroundings, which may consist of multiple surfaces at various temperatures, is incident upon the surface. The surface might also be irradiated by the sun or by a laser. In any case, we define the *irradiation*, G (W/m^2), as the rate at which radiation is incident upon the surface per unit surface area, over all wavelengths and from all directions. The two remaining heat fluxes of Table 6.1 are readily described once we consider the fate of the irradiation arriving at the surface. When radiation is incident upon a semitransparent medium, portions of the irradiation may be *reflected*, *absorbed*, and *transmitted* illustrated in Figure 6.3.a. Transmission, τ, refers to radiation passing through the medium, as occurs when a layer of water or a glass plate is irradiated by the sun or artificial lighting.

Absorption occurs when radiation interacts with the medium, causing an increase in the internal thermal energy of the medium. Reflection is the process of incident radiation being redirected away from the surface, with no effect on the medium. We define reflectivity, ρ, as the fraction of the irradiation that is reflected, absorptivity, α, as the fraction of the irradiation that is absorbed, and transmissivity, τ, as the fraction of the irradiation that is transmitted. Because all of the *irradiation* must be *reflected*, *absorbed*, or *transmitted*, it follows that

$$\rho + \alpha + \tau = 1 \qquad (6\text{-}2)$$

A medium that experiences no transmission (τ = 0) is opaque, in which case

$$\rho + \alpha = 1 \qquad (6\text{-}3)$$

With this understanding of the partitioning of the irradiation into reflected, absorbed, and transmitted components, two additional and useful radiation fluxes can be defined. The *radiosity*, J (W/m^2), of a surface accounts for all the radiant energy leaving the surface. For an opaque surface, it includes emission and the reflected portion of the irradiation, as illustrated in Figure 6.3.b. It is therefore expressed as

$$J = E + G_{ref} = E + \rho G \qquad (6\text{-}4)$$

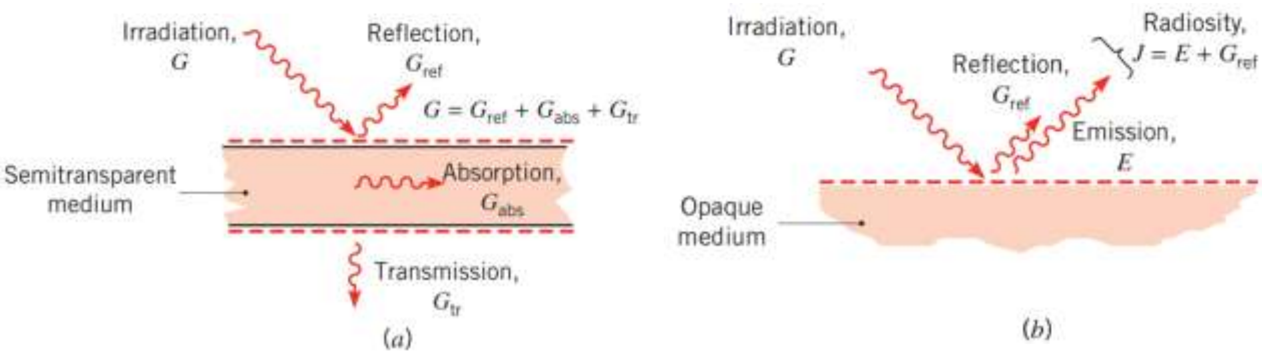


Figure 6.3: Radiation at a surface. (**a**) Reflection, absorption, and transmission of irradiation for a semitransparent medium. (**b**) The radiosity for an opaque medium.

Radiosity can also be defined at a surface of a semitransparent medium. In that case, the radiosity leaving the top surface of Figure 6.3.a (not shown) would include radiation transmitted through the medium from below.

Finally, the *net radiative flux from a surface*, (W/m^2), is the difference between the outgoing and incoming radiation

$$q''_{rad} = J - G \quad (6\text{-}5)$$

Combining Equations 6.5, 6.4, and 6.3, the *net flux* for an opaque surface is

$$q''_{rad} = E + \rho G - G = \varepsilon \sigma T_s^4 - \alpha G \quad (6\text{-}6)$$

A similar expression may be written for a semitransparent surface involving the transmissivity. Because it affects the temperature distribution within the system, the *net radiative flux* (or net radiation heat transfer rate, $q'' = q''_{rad}\, A$), is an important quantity in heat transfer analysis. As will become evident, the quantities *E*, *G*, and *J* are typically used to determine q''_{rad}, but they are also intrinsically important in applications involving *radiation detection* and *temperature measurement*.

6.2.3 Blackbody Radiation

To evaluate the *emissive power*, *irradiation*, *radiosity*, or *net radiative heat flux* of a real opaque surface, to do so, it is useful to first introduce the concept of a *blackbody*.

- ✓ A *blackbody* absorbs all incident radiation, regardless of wavelength and direction.
- ✓ For a prescribed temperature and wavelength, no surface can emit more energy than a blackbody.
- ✓ Although the radiation emitted by a blackbody is a function of wavelength and temperature, it is independent of direction. That is, the blackbody is a diffuse emitter.

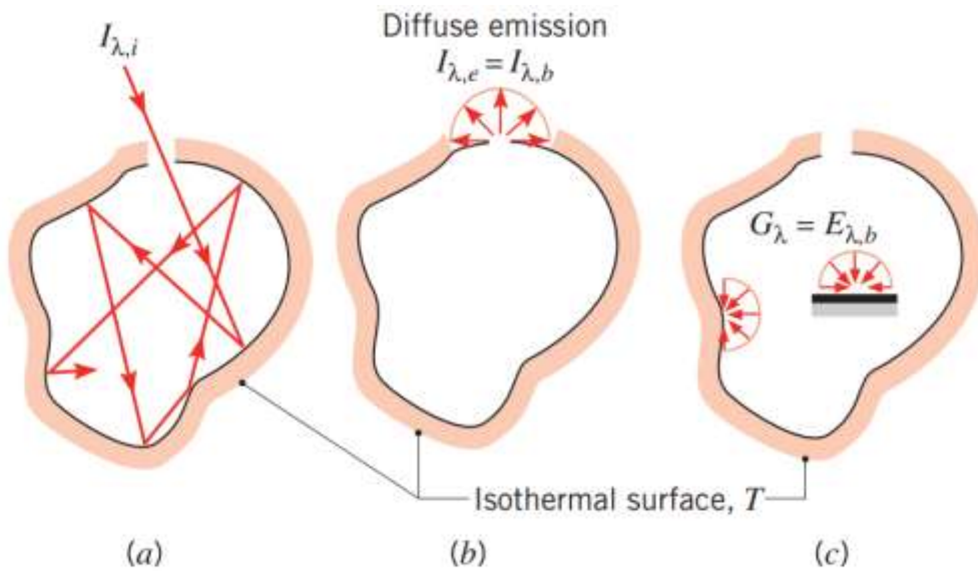


Figure 6.4: Characteristics of an isothermal blackbody cavity. (**a**) Complete absorption. (**b**) Diffuse emission from an aperture. (**c**) Diffuse irradiation of interior surfaces.

As the perfect absorber and emitter, the *blackbody* serves as a standard against which the *radiative properties* of actual surfaces may be compared.

Although closely approximated by some surfaces, it is important to note that no surface has precisely the properties of a blackbody. The closest approximation is achieved by a cavity whose inner surface is at a uniform temperature. If radiation enters the cavity through a small aperture (Figure 6.4.a), it is likely to experience many reflections before reemergence. Since some radiation is absorbed by the inner surface upon each reflection, it is eventually almost entirely absorbed by the cavity, and blackbody behavior is approximated. From thermodynamic principles it may then be argued that radiation leaving the aperture depends only on the surface temperature and corresponds to blackbody emission (Figure 6.4.b). Since blackbody emission is diffuse, the spectral intensity $I_{\lambda,b}$ of radiation leaving the cavity is independent of direction. Moreover, since the radiation field in the cavity, which is the cumulative effect of emission and reflection from the cavity surface, must be of the same form as the radiation emerging from the aperture, it also follows that a blackbody radiation field exists within the cavity. Accordingly, any small surface in the cavity (Figure 6.4.c) experiences irradiation for which $G_\lambda = E_{\lambda,b}(\lambda, T)$. This surface is *diffusely irradiated*, regardless of its orientation. *Blackbody radiation exists within the cavity irrespective of whether the cavity surface is highly reflecting or absorbing*.

6.3 Radiation Exchange between Surfaces

6.3.1 The View Factor

A. The View Factor Integral

The view factor F_{ij} is defined as the *fraction of the radiation leaving surface i that is intercepted by surface j*. To develop a general expression for F_{ij}, we consider the arbitrarily oriented surfaces A_i and A_j of Figure 6.5. Elemental areas on each surface, dA_i and dA_j, are connected by a line of length R, which forms the polar angles θ_i and θ_j, respectively, with the surface normals $\mathbf{n}_i$ and $\mathbf{n}_j$. The values of R, θ_i, and θ_j vary with the position of the elemental areas on A_i and A_j.

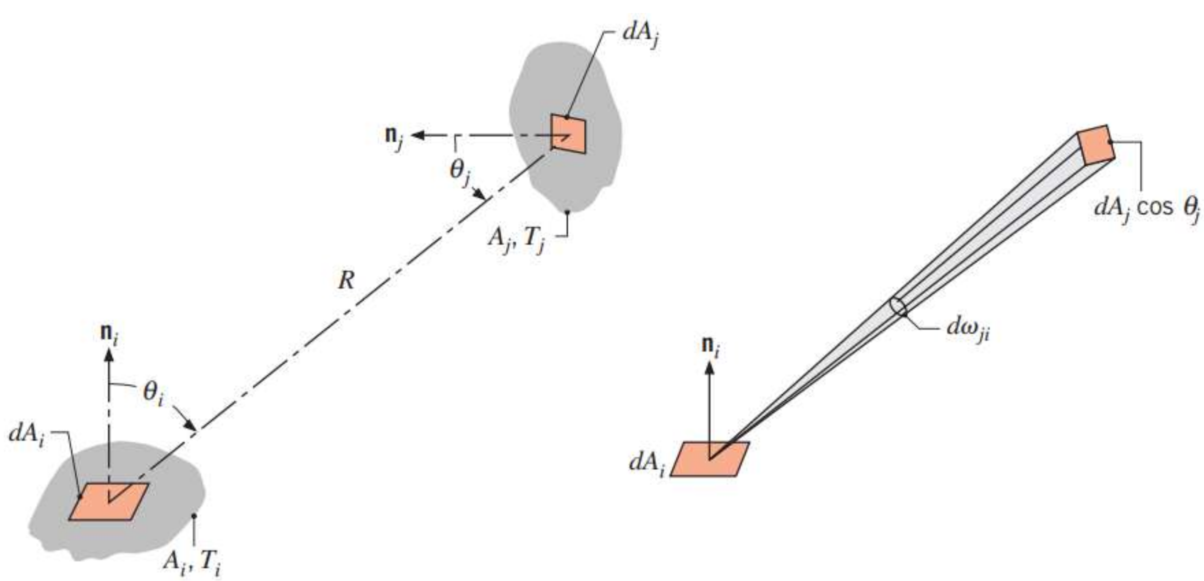


Figure 6.5: View factor associated with radiation exchange between elemental surfaces of area dA_i and dA_j.

From the definition of the *radiation intensity*, the rate at which radiation *leaves* dA_i and is *intercepted* by dA_j may be expressed as

$$dq_{i\to j} = I_{e+r,i} \cos\theta_i \, dA_i d\omega_{j-i} \quad (6\text{-}7)$$

where $I_{e+r,i}$ is the intensity of radiation leaving surface *i* by emission and reflection and $d\omega_{j-i}$ is the solid angle subtended by dA_j when viewed from dA_i. With $d\omega_{j-i} = (\cos\theta_j \, dA_j)/R^2$, it follows that

$$dq_{i\to j} = I_{e+r,i} \frac{\cos\theta_i \cos\theta_j}{R^2} dA_i dA_j \quad (6\text{-}8)$$

Assuming that surface *i* emits and reflects diffusely, we then obtain

$$dq_{i\to j} = J_i \frac{\cos\theta_i \cos\theta_j}{\pi R^2} dA_i dA_j \quad (6\text{-}9)$$

The total rate at which radiation leaves surface *i* and is intercepted by *j* may then be obtained by integrating over the two surfaces. That is,

$$dq_{i\to j} = J_i \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} dA_i dA_j \quad (6\text{-}10)$$

where it is assumed that the *radiosity* J_i is uniform over the surface A_i. From the definition of the view factor as the fraction of the radiation that leaves A_i and is intercepted by A_j,

$$F_{ij} = \frac{q_{i\to j}}{A_i J_i} \quad (6\text{-}11)$$

it follows that

$$F_{ij} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} dA_i dA_j \quad (6\text{-}12)$$

Similarly, the view factor F_{ji} is defined as the fraction of the radiation that leaves A_j and is intercepted by A_i. The same development then yields

$$F_{ji} = \frac{1}{A_j} \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} dA_i dA_j \quad (6\text{-}13)$$

Either Equation 6.12 or 6.13 may be used to determine the *view factor* associated with any two surfaces that are *diffuse emitters and reflectors* and have *uniform radiosity*.

B. Relations between Shape Factors

i. Reciprocity relation

An important view factor relation is suggested by Equations 6.12 and 6.13. In particular, equating the integrals appearing in these equations, it follows that

$$A_i F_{ij} = A_j F_{ji} \tag{6-14}$$

This expression, termed the *reciprocity relation*, is useful in determining one view factor from knowledge of the other.

ii. Summation rule

Another important view factor relation pertains to the surfaces of an enclosure (Figure 6.6). From the definition of the view factor, the *summation rule*

$$\sum_{j=1}^{N} F_{ij} = 1 \tag{6-15}$$

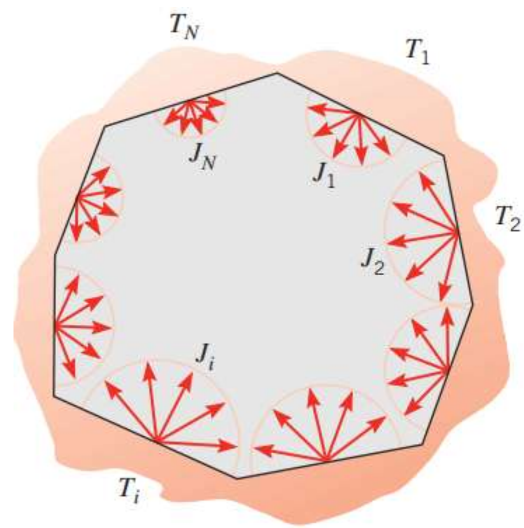
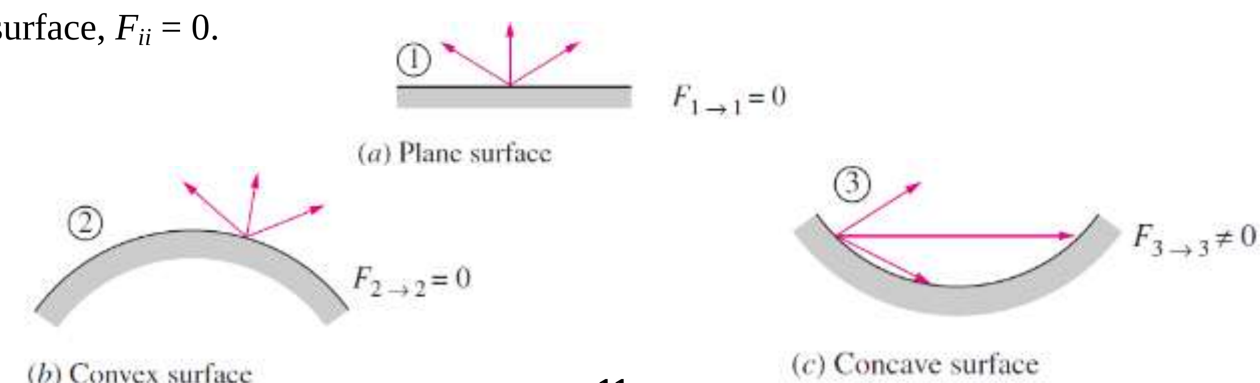


Figure 6.6: Radiation exchange in an enclosure.

may be applied to each of the *N* surfaces in the enclosure. This rule follows from the conservation requirement that all radiation leaving surface *i* must be intercepted by the enclosure surfaces. The term F_{ii} appearing in this summation represents the fraction of the radiation that leaves surface *i* and is directly intercepted by *i*. If the surface is concave, it sees itself and F_{ii} is *nonzero*. However, for a plane or convex surface, $F_{ii} = 0$.



A total of N view factors may be obtained from the N equations associated with application of the *summation rule*, Equation 6.15, to each of the surfaces in the enclosure. In addition, $N(N-1)/2$ view factors may be obtained from the $N(N-1)/2$ applications of the reciprocity relation, Equation 6.14, which are possible for the enclosure. Accordingly, only $[N^2-N-N(N-1)/2]=N(N-1)/2$ view factors need be determined directly. *For example*, in a three-surface enclosure this requirement corresponds to only 3(3-1)/2= 3 view factors. The remaining *six* view factors may be obtained by solving the *six* equations that result from use of Equations 6.14 and 6.15.

To illustrate the foregoing procedure, consider a simple, two-surface enclosure involving the spherical surfaces of Figure 6.7. Although the enclosure is characterized by N_2=4 view factors (F_{11}, F_{12}, F_{21}, F_{22}), only $N(N-1)/2=1$ view factor need be determined directly. In this case such a determination may be made by *inspection*. In particular, since all radiation leaving the inner surface must reach the outer surface, it follows that $F_{12}=1$. The same may not be said of radiation leaving the outer surface, since this surface sees itself. However, from the *reciprocity relation*, Equation 6.14, we obtain

$$F_{21} = \left(\frac{A_1}{A_2}\right) F_{12} = \left(\frac{A_1}{A_2}\right) \quad (6\text{-}16)$$

From the summation rule, we also obtain

$$F_{11} + F_{12} = 1$$

in which case F_{11}= 0, and i

$$F_{21} + F_{22} = 1$$

in which case

$$F_{22} = 1 - \left(\frac{A_1}{A_2}\right)$$

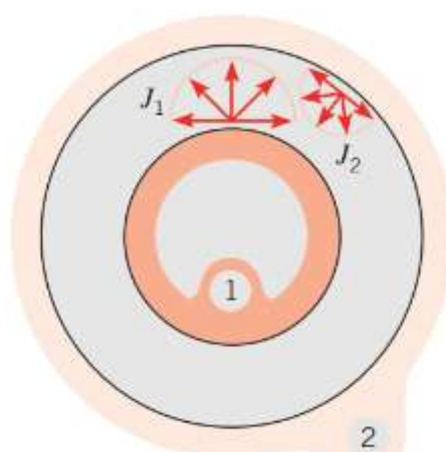


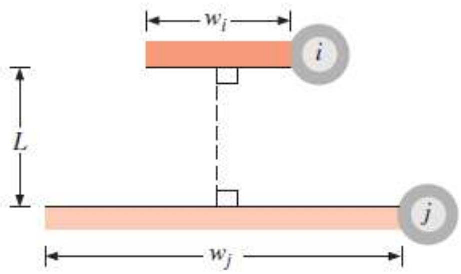
Figure 6.7: View factors for the enclosure formed by two spheres.

The view factors for several common geometries are presented in Tables 6.1 and 6.2. The configurations of Table 6.1 are assumed to be infinitely long (in a direction perpendicular to the page) and are hence *two-dimensional*. The configurations of Table 6.2 are three-dimensional.

Table 6.1: View Factors for *Two-Dimensional* Geometries.

Geometry — **Relation**

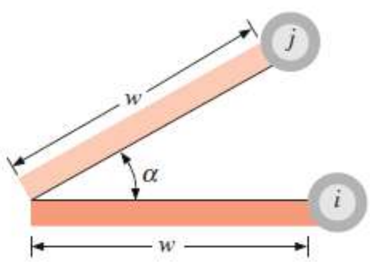
Parallel Plates with Midlines Connected by Perpendicular



$$F_{ij} = \frac{[(W_i + W_j)^2 + 4]^{1/2} - [(W_j - W_i)^2 + 4]^{1/2}}{2W_i}$$

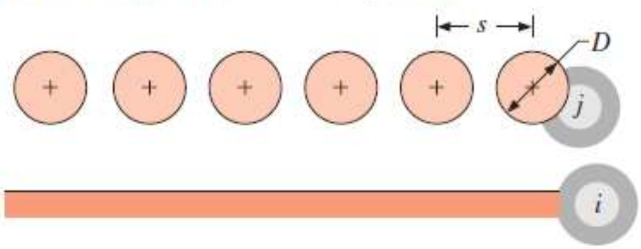
$$W_i = w_i/L,\ W_j = w_j/L$$

Inclined Parallel Plates of Equal Width and a Common Edge



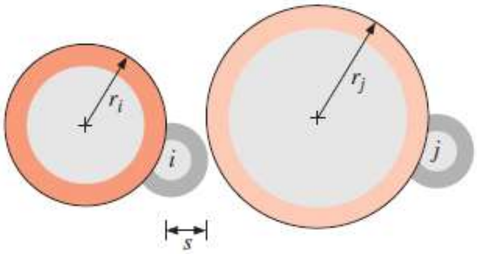
$$F_{ij} = 1 - \sin\left(\frac{\alpha}{2}\right)$$

Innite Plane and Row of Cylinders



$$F_{ij} = 1 - \left[1 - \left(\frac{D}{s}\right)^2\right]^{1/2} + \left(\frac{D}{s}\right)\tan^{-1}\left[\left(\frac{s^2 - D^2}{D^2}\right)^{1/2}\right]$$

Parallel Cylinders of Different Radii

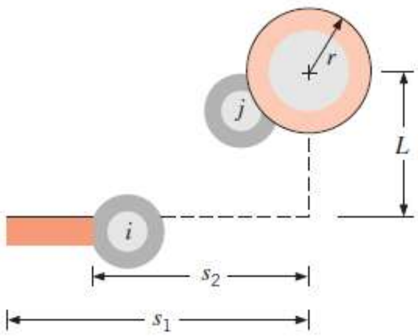


$$F_{ij} = \frac{1}{2\pi}\left\{\pi + [C^2 - (R+1)^2]^{1/2} - [C^2 - (R-1)^2]^{1/2} + (R-1)\cos^{-1}\left[\left(\frac{R}{C}\right) - \left(\frac{1}{C}\right)\right] - (R+1)\cos^{-1}\left[\left(\frac{R}{C}\right) + \left(\frac{1}{C}\right)\right]\right\}$$

$R = r_j/r_i$, $S = s/r_i$

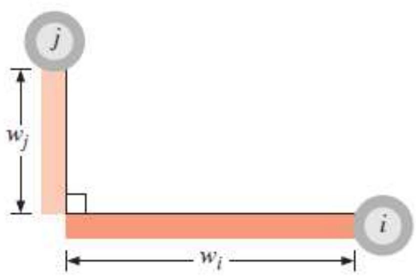
$C = 1 + R + S$

Cylinder and Parallel Rectangle



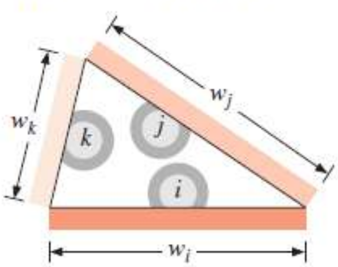
$$F_{ij} = \frac{r}{s_1 - s_2}\left[\tan^{-1}\frac{s_1}{L} - \tan^{-1}\frac{s_2}{L}\right]$$

Perpendicular Plates with a Common Edge



$$F_{ij} = \frac{1 + (w_j/w_i) - [1 + (w_j/w_i)^2]^{1/2}}{2}$$

Three-Sided Enclosure

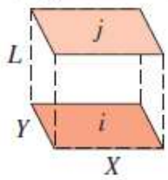


$$F_{ij} = \frac{w_i + w_j - w_k}{2w_i}$$

Geometry	Relation

Aligned Parallel Rectangles

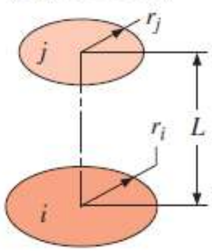
(Figure 13.4)



$$\overline{X} = X/L,\ \overline{Y} = Y/L$$

$$F_{ij} = \frac{2}{\pi \overline{X}\,\overline{Y}} \Bigg\{ \ln \left[\frac{(1+\overline{X}^2)(1+\overline{Y}^2)}{1+\overline{X}^2+\overline{Y}^2} \right]^{1/2} + \overline{X}(1+\overline{Y}^2)^{1/2} \tan^{-1} \frac{\overline{X}}{(1+\overline{Y}^2)^{1/2}} + \overline{Y}(1+\overline{X}^2)^{1/2} \tan^{-1} \frac{\overline{Y}}{(1+\overline{X}^2)^{1/2}} - \overline{X}\tan^{-1}\overline{X} - \overline{Y}\tan^{-1}\overline{Y} \Bigg\}$$

Coaxial Parallel Disks
(Figure 13.5)

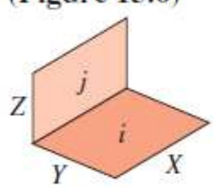


$$R_i = r_i/L,\ R_j = r_j/L$$

$$S = 1 + \frac{1+R_j^2}{R_i^2}$$

$$F_{ij} = \frac{1}{2}\{S - [S^2 - 4(r_j/r_i)^2]^{1/2}\}$$

Perpendicular Rectangles with a Common Edge
(Figure 13.6)



$$H = Z/X,\ W = Y/X$$

$$F_{ij} = \frac{1}{\pi W} \Bigg(W \tan^{-1}\frac{1}{W} + H \tan^{-1}\frac{1}{H} - (H^2+W^2)^{1/2} \tan^{-1} \frac{1}{(H^2+W^2)^{1/2}} + \frac{1}{4} \ln \Bigg\{ \frac{(1+W^2)(1+H^2)}{1+W^2+H^2} \left[\frac{W^2(1+W^2+H^2)}{(1+W^2)(W^2+H^2)} \right]^{W^2} \times \left[\frac{H^2(1+H^2+W^2)}{(1+H^2)(H^2+W^2)} \right]^{H^2} \Bigg\} \Bigg)$$

The view factors for several common geometries are presented in Figures 6.8 through 6.11.

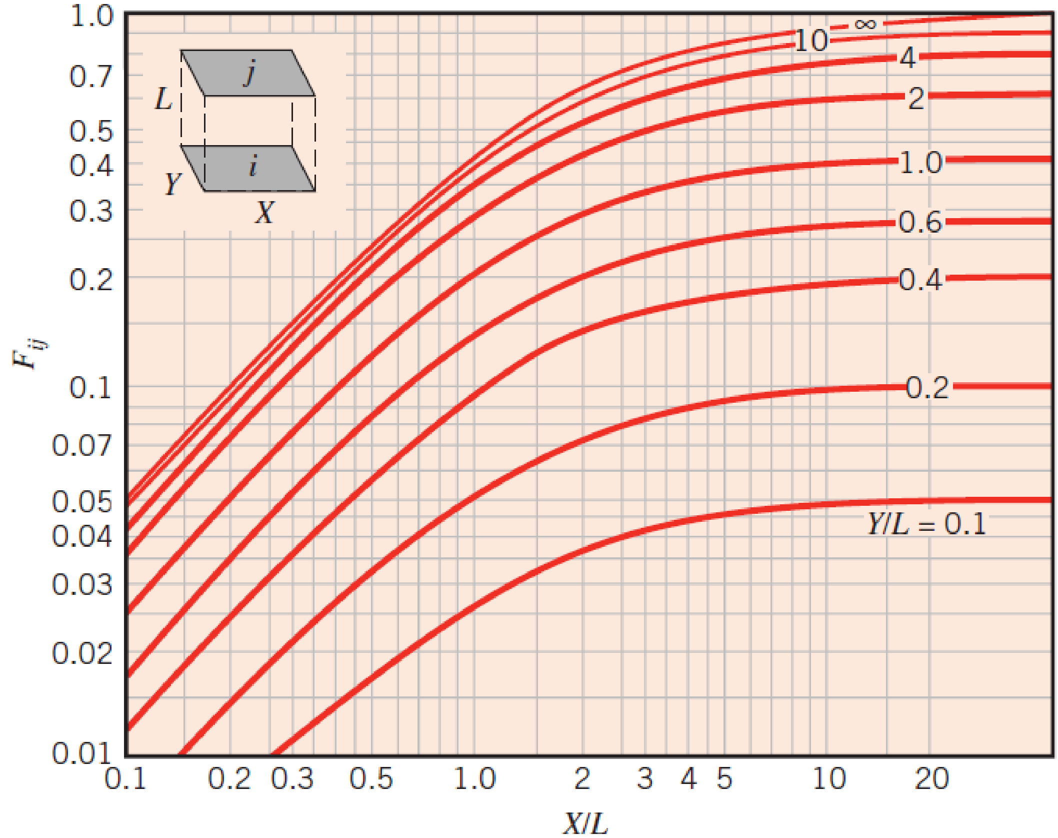
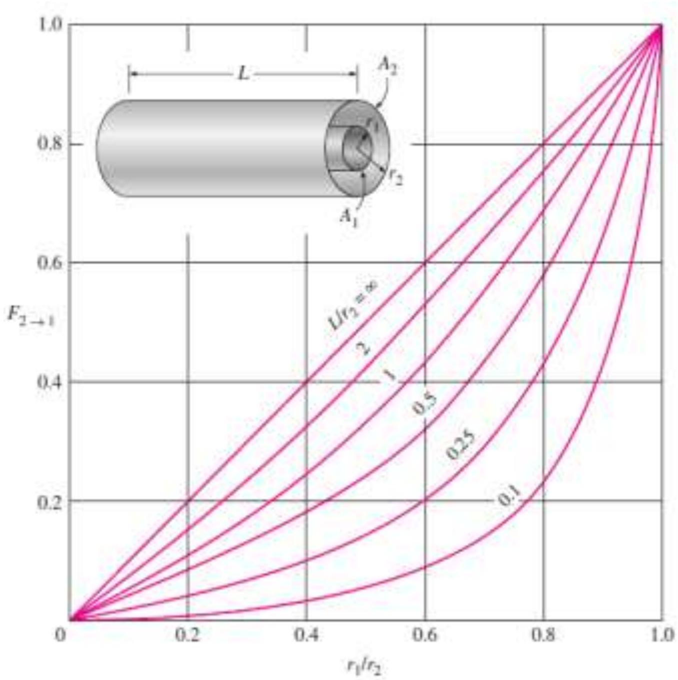


Figure 6.8: View factor for aligned parallel rectangles.



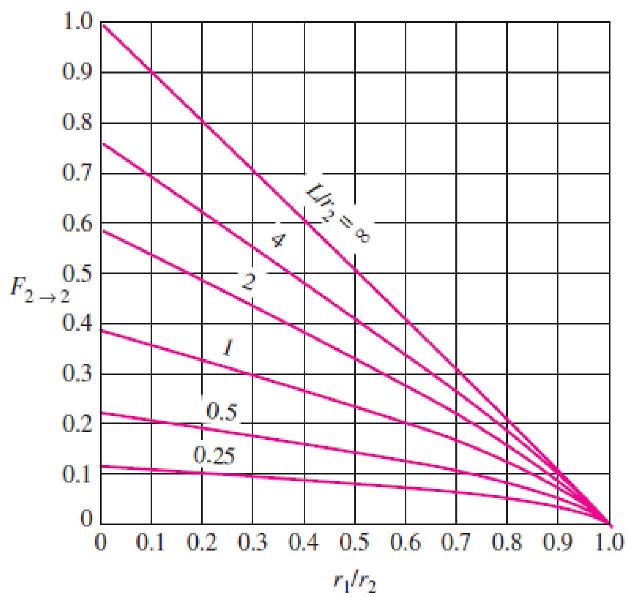


Figure 6.9: View factors for two concentric cylinders of finite length: (a) outer cylinder to inner cylinder; (b) outer cylinder to itself.

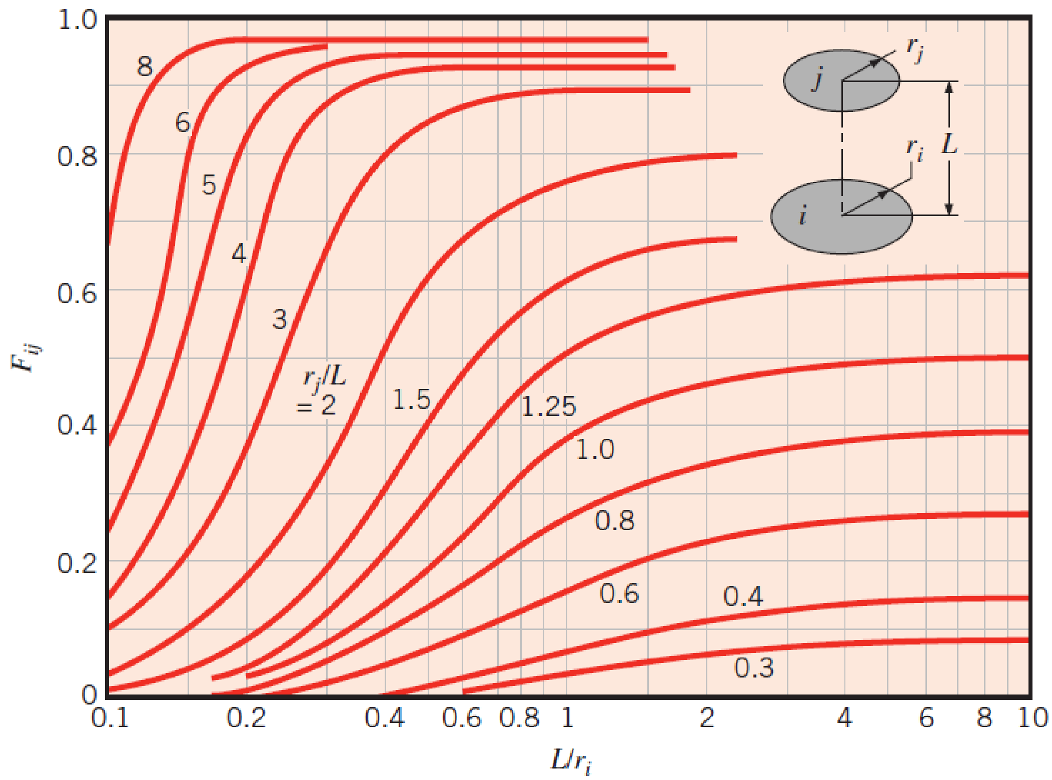


Figure 6.10: View factor for coaxial parallel disks.

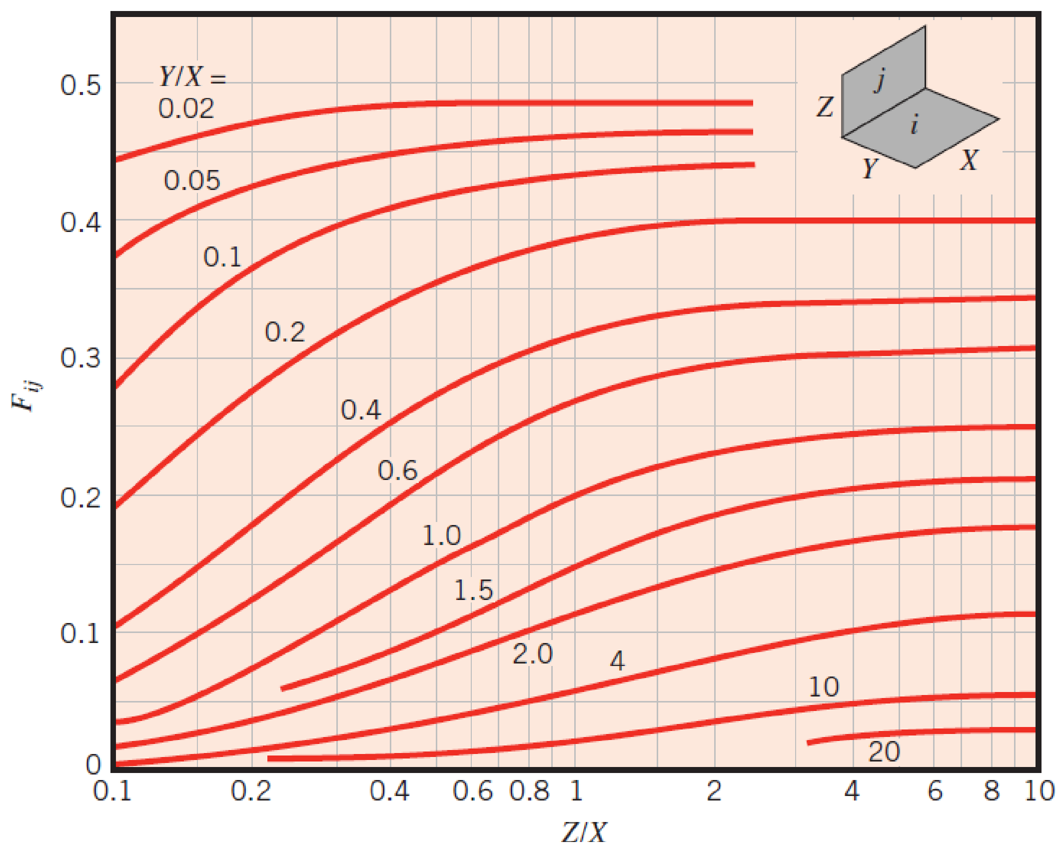


Figure 6.11: View factor for perpendicular rectangles with a common edge.

iii. The Superposition Rule

The first relation concerns the additive nature of the view factor for a subdivided surface and may be inferred from Figure 6.12. Considering radiation from surface *i* to surface *j*, which is divided into *n* components, it is evident that

$$F_{i(j)} = \sum_{k=1}^{n} F_{ik} \quad (6\text{-}17)$$

where the parentheses around a subscript indicate that it is a composite surface, in which case (*j*) is equivalent to (1, 2, . . . , *k*, . . . , *n*). This expression simply states that radiation reaching a composite surface is the sum of the radiation reaching its parts. Although it pertains to subdivision of the receiving surface, it may also be used to obtain the second view factor relation, which pertains to subdivision of the originating surface. Multiplying Equation 6.17 by A_i and applying the reciprocity relation, Equation 6.14, to each of the resulting terms, it follows that

$$A_j F_{(j)i} = \sum_{k=1}^{n} A_k F_{ki} \quad (6\text{-}18)$$

$$F_{(j)i} = \frac{\sum_{k=1}^{n} A_k F_{ki}}{\sum_{k=1}^{n} A_k} \quad (6\text{-}19)$$

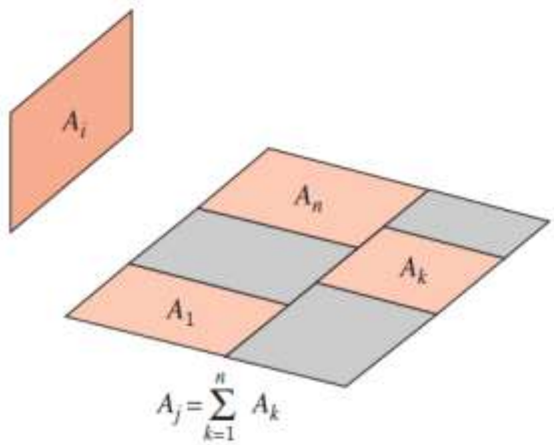
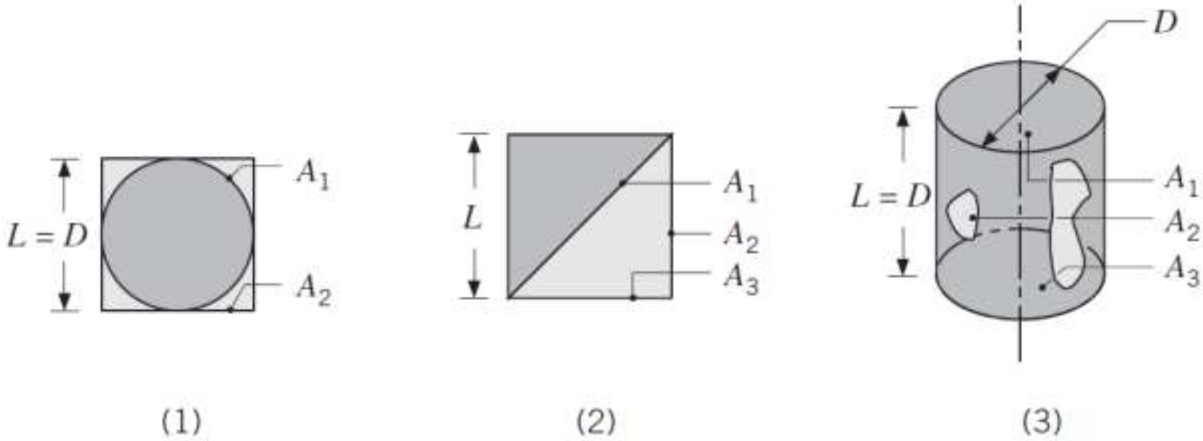


Figure 6.12: Areas used to illustrate view factor relations.

Equations 6.18 and 6.19 may be applied when the originating surface is composed of several parts.

Example 1: Determine the view factors F_{12} and F_{21} for the following geometries:



1. Sphere of diameter D inside a cubical box of length $L = D$.
2. One side of a diagonal partition within a long square duct.
3. End and side of a circular tube of equal length and diameter.

Solution:

1. Sphere within a cube:
By inspection, $F_{12} = 1$

By reciprocity, $F_{21} = \dfrac{A_1}{A_2} F_{12} = \dfrac{\pi D^2}{6L^2} \times 1 = \dfrac{\pi}{6}$

2. Partition within a square duct:
From summation rule, $F_{11} + F_{12} + F_{13} = 1$
where $F_{11} = 0$
By symmetry, $F_{12} = F_{13}$
Hence $F_{12} = 0.50$

By reciprocity, $F_{21} = \dfrac{A_1}{A_2} F_{12} = \dfrac{\sqrt{2}L}{L} \times 0.5 = 0.71$

3. Circular tube:
From Table 13.2 or Figure 13.5, with $(r_3/L) = 0.5$ and $(L/r_1) = 2$, $F_{13} = 0.172$
From summation rule, $F_{11} + F_{12} + F_{13} = 1$
or, with $F_{11} = 0$, $F_{12} = 1 - F_{13} = 0.828$

From reciprocity, $F_{21} = \dfrac{A_1}{A_2} F_{12} = \dfrac{\pi D^2/4}{\pi DL} \times 0.828 = 0.207$

Example 2: Determine the view factors from the base of the pyramid shown in Figure 6.13 to each of its four side surfaces. The base of the pyramid is a square, and its side surfaces are isosceles triangles.

Solution: The base of the pyramid (surface 1) and its four side surfaces (surfaces 2, 3, 4, and 5) form a five-surface enclosure. The first thing we notice about this enclosure is its symmetry. The four side surfaces are symmetric about the base surface. Then, from the *symmetry rule*, we have

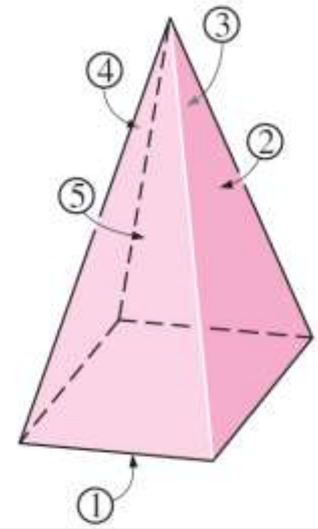


Figure 6.13: The pyramid considered in Example 2.

$$F_{12} = F_{13} = F_{14} = F_{15}$$

Also, the *summation rule* applied to surface 1 yields

$$\sum_{j=1}^{5} F_{1j} = F_{11} + F_{12} + F_{13} + F_{14} + F_{15} = 1$$

However, $F_{11} = 0$, since the base is a *flat* surface. Then the two relations above yield

$$F_{12} = F_{13} = F_{14} = F_{15} = \mathbf{0.25}$$

Example 3: Determine the view factor from any one side to any other side of the infinitely long triangular duct whose cross section is given in Figure 6.14.

Solution: The view factors associated with an infinitely long triangular duct are to be determined.

The widths of the sides of the triangular cross section of the duct are L_1, L_2, and L_3, and the surface areas corresponding to them are A_1, A_2, and A_3, respectively. Since the duct is infinitely long, the fraction of radiation leaving any surface that escapes through the ends of the duct is negligible. Therefore, the infinitely long duct can be considered to be a three-surface enclosure, $N = 3$.

This enclosure involves $N^2 = 3^2 = 9$ view factors, and we need to determine

$$\tfrac{1}{2}N(N-1) = \tfrac{1}{2} \times 3(3-1) = 3$$

of these view factors directly. Fortunately, we can determine all three of them by inspection to be

$$F_{11} = F_{22} = F_{33} = 0$$

since all three surfaces are flat. The remaining six view factors can be determined by the application of the summation and reciprocity rules.

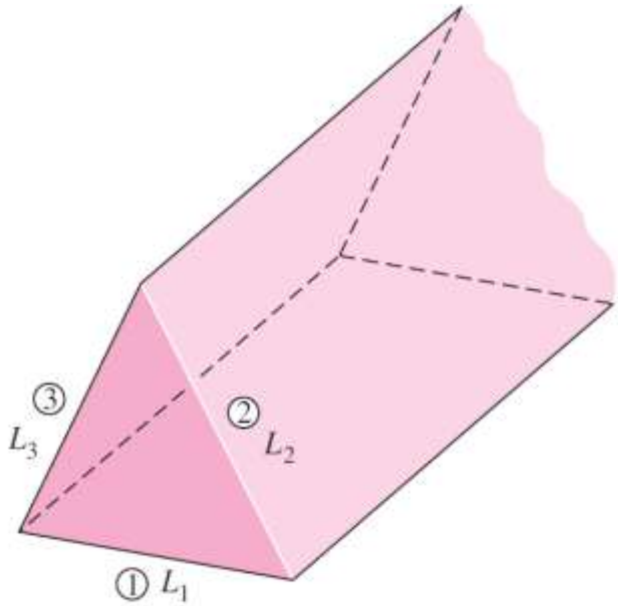


Figure 6.14: The infinitely long triangular duct considered in Example 3.

Applying the summation rule to each of the three surfaces gives

$$\begin{aligned} F_{11} + F_{12} + F_{13} &= 1 \\ F_{21} + F_{22} + F_{23} &= 1 \\ F_{31} + F_{32} + F_{33} &= 1 \end{aligned}$$

Noting that $F_{11} = F_{22} = F_{33} = 0$ and multiplying the first equation by A_1, the second by A_2, and the third by A_3 gives

$$\begin{aligned} A_1F_{12} + A_1F_{13} &= A_1 \\ A_2F_{21} + A_2F_{23} &= A_2 \\ A_3F_{31} + A_3F_{32} &= A_3 \end{aligned}$$

Finally, applying the three reciprocity relations $A_1F_{12} = A_2F_{21}$, $A_1F_{13} = A_3F_{31}$, and $A_2F_{23} = A_3F_{32}$ gives

$$\begin{aligned} A_1F_{12} + A_1F_{13} &= A_1 \\ A_1F_{12} + A_2F_{23} &= A_2 \\ A_1F_{13} + A_2F_{23} &= A_3 \end{aligned}$$

This is a set of three algebraic equations with three unknowns, which can be solved to obtain

$$F_{12} = \frac{A_1 + A_2 - A_3}{2A_1} = \frac{L_1 + L_2 - L_3}{2L_1}$$

$$F_{13} = \frac{A_1 + A_3 - A_2}{2A_1} = \frac{L_1 + L_3 - L_2}{2L_1}$$

$$F_{23} = \frac{A_2 + A_3 - A_1}{2A_2} = \frac{L_2 + L_3 - L_1}{2L_2}$$

6.3.2 Blackbody Radiation Exchange

In general, radiation may leave a surface due to both reflection and emission, and on reaching a second surface, experience reflection as well as absorption. However, matters are simplified for surfaces that may be approximated as blackbodies, since there is no reflection. Hence energy leaves only as a result of emission, and all incident radiation is absorbed. Consider radiation exchange between two black surfaces of arbitrary shape (Figure 6.15). Defining $q_{i\text{-}j}$ as the rate at which radiation *leaves* surface *i* and is *intercepted* by surface *j*, it follows that

$$q_{i\rightarrow j} = (A_i J_i)\, F_{i\rightarrow j} \qquad (6\text{-}20)$$

or, since *radiosity* equals *emissive power* for a black surface ($J_i = E_{bi}$),

$$q_{i\rightarrow j} = A_i E_{bi}\, F_{i\rightarrow j} \qquad (6\text{-}21)$$

Similarly,

$$q_{j\rightarrow i} = A_j E_{bj}\, F_{j\rightarrow i} \qquad (6\text{-}22)$$

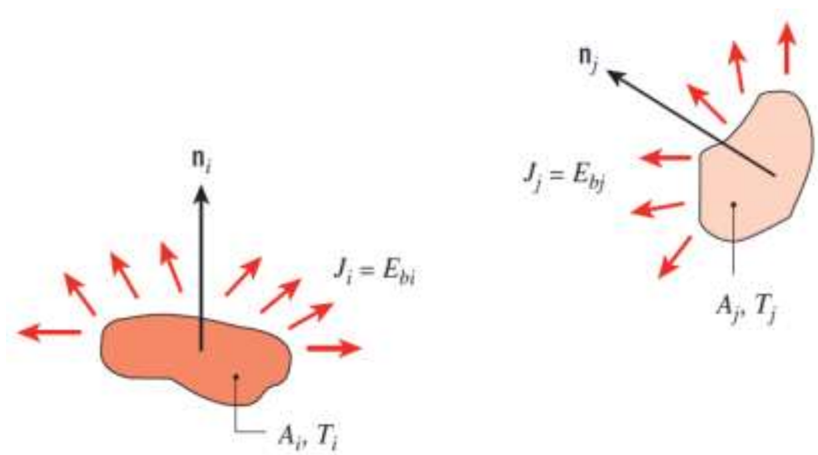


Figure 6.15: Radiation transfer between two surfaces that may be approximated as blackbodies.

The *net radiative exchange* between the two surfaces may then be defined as

$$q_{ij} = q_{i\to j} - q_{j\to i} \tag{6-23}$$

from which it follows that

$$q_{ij} = A_i E_{bi} F_{i\to j} - A_j E_{bj} F_{j\to i} \tag{6-24}$$

$$q_{ij} = A_i F_{i\to j}\sigma\left(T_i^4 - T_j^4\right) \tag{6-25}$$

Equation 6.25 provides the *net* rate at which radiation *leaves* surface *i* as a result of its interaction with *j*, which is equal to the *net* rate at which *j gains* radiation due to its interaction with *i*.

The foregoing result may also be used to evaluate the net radiation transfer from any surface in an *enclosure* of black surfaces. With N surfaces maintained at different temperatures, the net transfer of radiation from surface *i* is due to exchange with the remaining surfaces and may be expressed as

$$q_i = \sum_{j=1}^{N} A_i F_{i\to j}\sigma\left(T_i^4 - T_j^4\right) \tag{6-26}$$

The net radiative heat flux, $q_i'' = q_i/A_i$, was denoted as q_{rad}'' in Chapter 1. The subscript *rad* has been dropped here for convenience.

Example 4: A furnace cavity, which is in the form of a cylinder of 50mm diameter and 150mm length, is open at one end to large surroundings (see Figure 6.16) that are at 27°C. The bottom of the cavity is heated independently, as are three annular sections that comprise the sides of the cavity. All interior surfaces of the cavity may be approximated as blackbodies and are maintained at 1650°C. What is the required electrical power input to the bottom surface of the cavity? What is the electrical power to the top, middle, and bottom sections of the cavity sides? The backs of the electrically heated surfaces are well insulated.

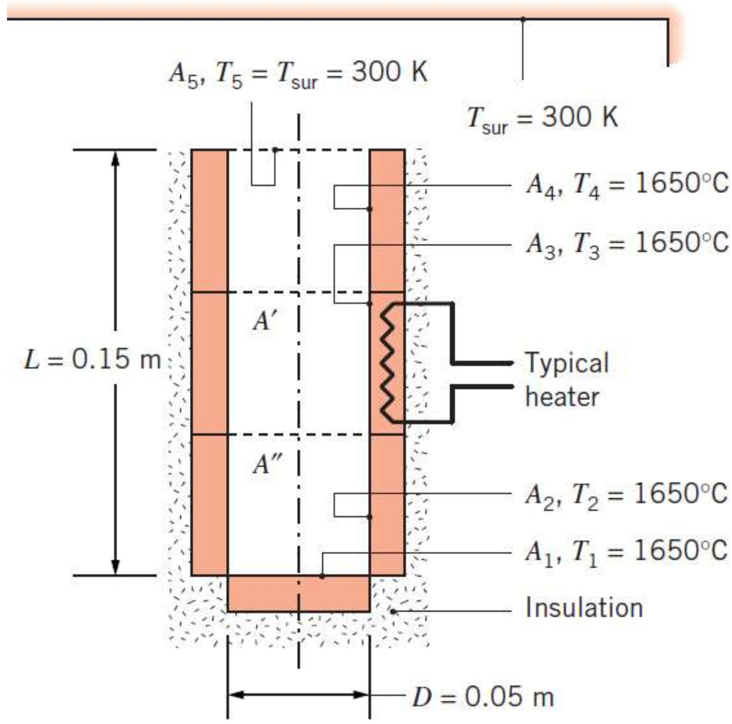


Figure 6.16: Temperature of furnace surfaces and surroundings.

Surface 1: $$q_1 = A_1F_{15}\sigma(T_1^4 - T_5^4) = A_5F_{51}\sigma(T_1^4 - T_5^4) \quad (1)$$

Surface 2: $$q_2 = A_2F_{25}\sigma(T_2^4 - T_5^4) = A_5F_{52}\sigma(T_2^4 - T_5^4) \quad (2)$$

Surface 3: $$q_3 = A_3F_{35}\sigma(T_3^4 - T_5^4) = A_5F_{53}\sigma(T_3^4 - T_5^4) \quad (3)$$

Surface 4: $$q_4 = A_4F_{45}\sigma(T_4^4 - T_5^4) = A_5F_{54}\sigma(T_4^4 - T_5^4) \quad (4)$$

We will determine the view factors by first defining two hypothetical surfaces A' and A'' as shown in the schematic. From Table 13.2 with $(r_i/L) = (r_j/L) = (0.025\text{ m}/0.150\text{ m}) = 0.167$, $F_{51} = 0.0263$. With $(r_i/L) = (r_j/L) = (0.025\text{ m}/0.100\text{ m}) = 0.25$, $F_{5A''} = 0.0557$ so that $F_{52} = F_{5A''} - F_{51} = 0.0557 - 0.0263 = 0.0294$. Likewise, with $(r_i/L) = (r_j/L) = (0.025\text{ m}/0.050\text{ m}) = 0.5$, $F_{5A'} = 0.172$ so that $F_{53} = F_{5A'} - F_{5A''} = 0.172 - 0.0557 = 0.1163$. Finally, $F_{54} = 1 - F_{5A'} = 1 - 0.172 = 0.828$. The electrical power delivered to each of the four furnace surfaces can now be determined by solving Equations 1 through 4 for the radiation loss from each surface with $A_5 = \pi D^2/4 = \pi \times (0.05\text{ m})^2/4 = 0.00196\text{ m}^2$.

$$q_1 = 0.00196\text{ m}^2 \times 0.0263 \times 5.67 \times 10^{-8}\text{ W/m}^2\cdot\text{K}^4 \times (1923\text{ K}^4 - 300\text{ K}^4) = 39.9\text{ W} \quad \triangleleft$$

$$q_2 = 0.00196\text{ m}^2 \times 0.0294 \times 5.67 \times 10^{-8}\text{ W/m}^2\cdot\text{K}^4 \times (1923\text{ K}^4 - 300\text{ K}^4) = 44.7\text{ W} \quad \triangleleft$$

$$q_3 = 0.00196\text{ m}^2 \times 0.1163 \times 5.67 \times 10^{-8}\text{ W/m}^2\cdot\text{K}^4 \times (1923\text{ K}^4 - 300\text{ K}^4) = 177\text{ W} \quad \triangleleft$$

$$q_4 = 0.00196\text{ m}^2 \times 0.828 \times 5.67 \times 10^{-8}\text{ W/m}^2\cdot\text{K}^4 \times (1923\text{ K}^4 - 300\text{ K}^4) = 1260\text{ W} \quad \triangleleft$$

6.3.3 Radiation Exchange between Opaque, Diffuse, Gray Surfaces in an Enclosure

In general, radiation may leave an opaque surface due to both *reflection* and *emission*, and on reaching a second opaque surface, experience reflection as well as absorption. In an enclosure, such as that of Figure 6.17.a, radiation may experience multiple reflections off all surfaces, with partial absorption occurring at each.

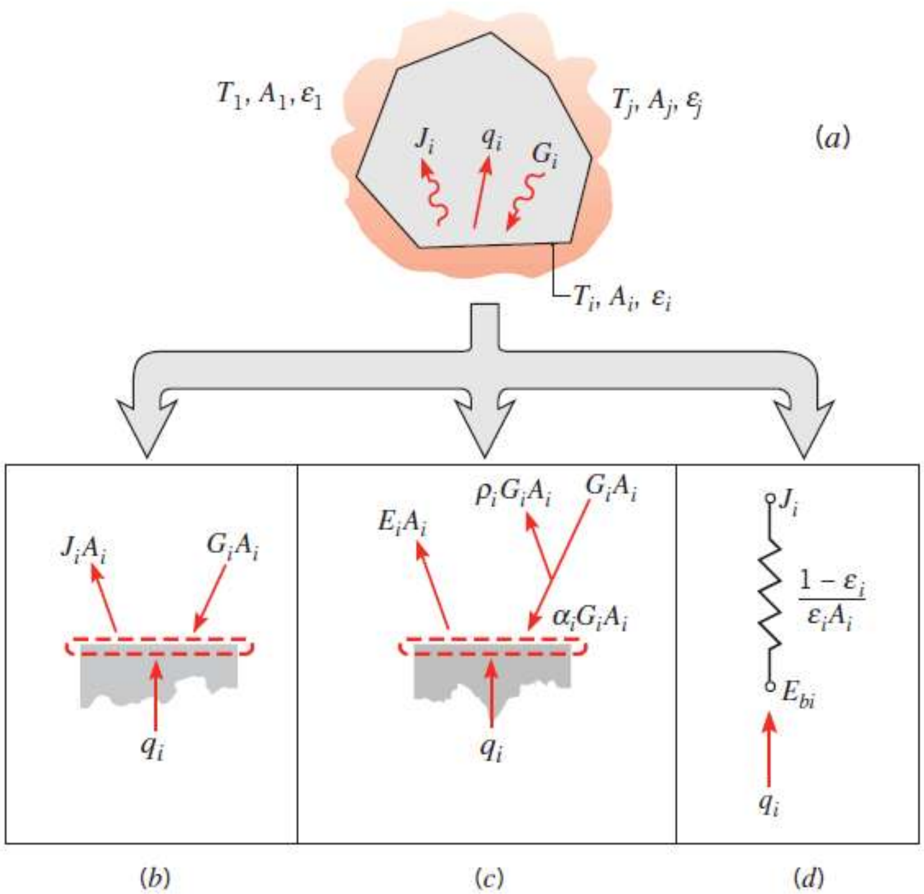


Figure 6.17: Radiation exchange in an enclosure of diffuse, gray surfaces with a nonparticipating medium. (a) Schematic of the enclosure. (b) Radiative balance. (c) Radiative balance. (d) Resistance representing net radiation transfer from a surface.

Analyzing radiation exchange in an enclosure may be simplified by making certain assumptions. Each surface of the enclosure is assumed to be *isothermal* and to be characterized by a *uniform radiosity* and a *uniform irradiation*. The surfaces are also assumed to be *opaque* (τ= 0) and to have emissivities, absorptivities, and reflectivities that are independent of direction (the surfaces are *diffuse*) *and* independent of wavelength (the surfaces are *gray*). Under these conditions the emissivity is equal to the absorptivity, $\varepsilon=\alpha$ (a form of Kirchhoff's law). Finally, the medium within the enclosure is taken to be *nonparticipating*. The problem is generally one in which either the temperature T_i or the net radiative heat flux q_i'' associated with each of the surfaces is known. The objective is to use this information to determine the unknown radiative heat fluxes and temperatures associated with each of the surfaces.

A. Net Radiation Exchange at a Surface

The term q_i, which is the *net* rate at which radiation *leaves* surface *i*, represents the net effect of radiative interactions occurring at the surface (Figure 6.17.b). It is the rate at which energy would have to be transferred to the surface by other means to maintain it at a constant temperature. It is equal to the difference between the surface radiosity and irradiation and may be expressed as

$$q_i = A_i(J_i - G_i) \quad (6\text{-}27)$$

Using the definition of the radiosity,

$$J_i = E_i + \rho_i G_i \quad (6\text{-}28)$$

The net radiative transfer from the surface may be expressed as

$$q_i = A_i(E_i - \alpha_i G_i) \quad (6\text{-}29)$$

where use has been made of the relationship $\alpha_i = 1 - \rho_i$ for an opaque surface. This relationship is illustrated in Figure 6.17.*c*. Noting that $E_i = \varepsilon_i E_{bi}$ and recognizing that $\rho_i = 1 - \alpha_i = 1 - \varepsilon_i$ for an opaque, diffuse, gray surface, the radiosity may also be expressed as

$$J_i = \varepsilon_i E_{bi} + (1 - \varepsilon_i)G_i \quad (6\text{-}30)$$

Solving for G_i and substituting into Equation 6.27, it follows that

$$q_i = A_i(J_i - \frac{J_i - \varepsilon_i E_{bi}}{1-\varepsilon_i}) \quad (6\text{-}31)$$

$$q_i = \frac{E_{bi} - J_i}{(1-\varepsilon_i)/\varepsilon_i A_i} \quad (6\text{-}32)$$

Equation 6.32 provides a convenient representation for the net radiative heat transfer rate from a surface. This transfer, which is represented in Figure 6.17.*d*, is associated with the driving potential (E_{bi} - J_i) and a *surface radiative resistance* of the form (1 - ε_i)/ ε_i A_i. Hence, if the emissive power that the surface would have if it were black exceeds its radiosity, there is net radiation heat transfer from the surface; if the inverse is true, the net transfer is to the surface.

It is sometimes the case that one of the surfaces is very large relative to the other surfaces under consideration. For example, the system might consist of multiple small surfaces in a large room. In this case, the area of the large surface is effectively infinite ($A_i \rightarrow \infty$), and we see that its surface radiative resistance, (1 - ε_i)/ ε_i A_i, is effectively zero, just as it would be for a black surface (ε_i =1). Hence, J_i = E_{bi}, and *a surface which is large relative to all other surfaces under consideration can be treated as if it were a blackbody*.

B. Radiation Exchange between Surfaces

To use Equation 6.32, the surface radiosity J_i must be known. To determine this quantity,it is necessary to consider radiation exchange between the surfaces of the enclosure. The irradiation of surface *i* can be evaluated from the radiosities of all the surfaces in the enclosure. In particular, from the definition of the view factor, it follows that the total rate at which radiation reaches surface *i* from all surfaces, including *i*, is

$$A_i G_i = \sum_{j=1}^{N} F_{ji} A_j J_j \quad (6\text{-}33)$$

or from the *reciprocity relation*, Equation 6.14,

$$A_i G_i = \sum_{j=1}^{N} F_{ij} A_i J_j \quad (6\text{-}34)$$

Canceling the area A_i and substituting into Equation 6.27 for G_i,

$$q_i = A_i(J_i - \sum_{j=1}^{N} F_{ij} J_j) \tag{6-35}$$

or, from the *summation rule*, Equation 6.15,

$$q_i = A_i(\sum_{j=1}^{N} F_{ij} J_i - \sum_{j=1}^{N} F_{ij} J_j) \tag{6-36}$$

Hence,

$$q_i = \sum_{j=1}^{N} F_{ij}\ A_i\left(J_i - J_j\right) = \sum_{j=1}^{N} q_{ij} \tag{6-37}$$

Radiation Network Approach

Equation 6.37equates the net rate of radiation transfer from surface *i*, *qi*, to the sum of components q_{ij} related to radiative exchange with the other surfaces. Each component may be represented by a *network element* for which (J_i - J_j) is the driving potential and $(A_iF_{ij})^{-1}$ is a *space* or *geometrical resistance* (Figure 6.18). Combining Equations 6.32 and 6.37, we then obtain

$$q_i = \sum_{j=1}^{N} \frac{(J_i - J_j)}{(A_i F_{ij})^{-1}} = \frac{E_{bi} - J_i}{(1-\varepsilon_i)/\varepsilon_i A_i} \tag{6-38}$$

As shown in Figure 6.18, this expression represents a radiation balance for the radiosity *node* associated with surface *i*. The rate of radiation transfer (current flow) to *i* through its surface resistance must equal the net rate of radiation transfer (current flows) from *i* to all other surfaces through the corresponding geometrical resistances.

Note that Equation 6.38 is especially useful when the surface temperature T_i (hence E_{bi}) is known. Although this situation is typical, it does not always apply. In particular, situations may arise for which the net radiation transfer rate at the surface q_i, rather than the temperature T_i, is known. In such cases the preferred form of the radiation balance is Equation 6.37, rearranged as

$$q_i = \sum_{j=1}^{N} \frac{(J_i - J_j)}{(A_i F_{ij})^{-1}} \tag{6-39}$$

Use of network representations was first suggested by Oppenheim. The network is built by first identifying nodes associated with the radiosities of each of the *N* surfaces of the enclosure. The method provides a useful tool for visualizing radiation exchange in the enclosure and, at least for simple enclosures, may be used as the basis for predicting this exchange.

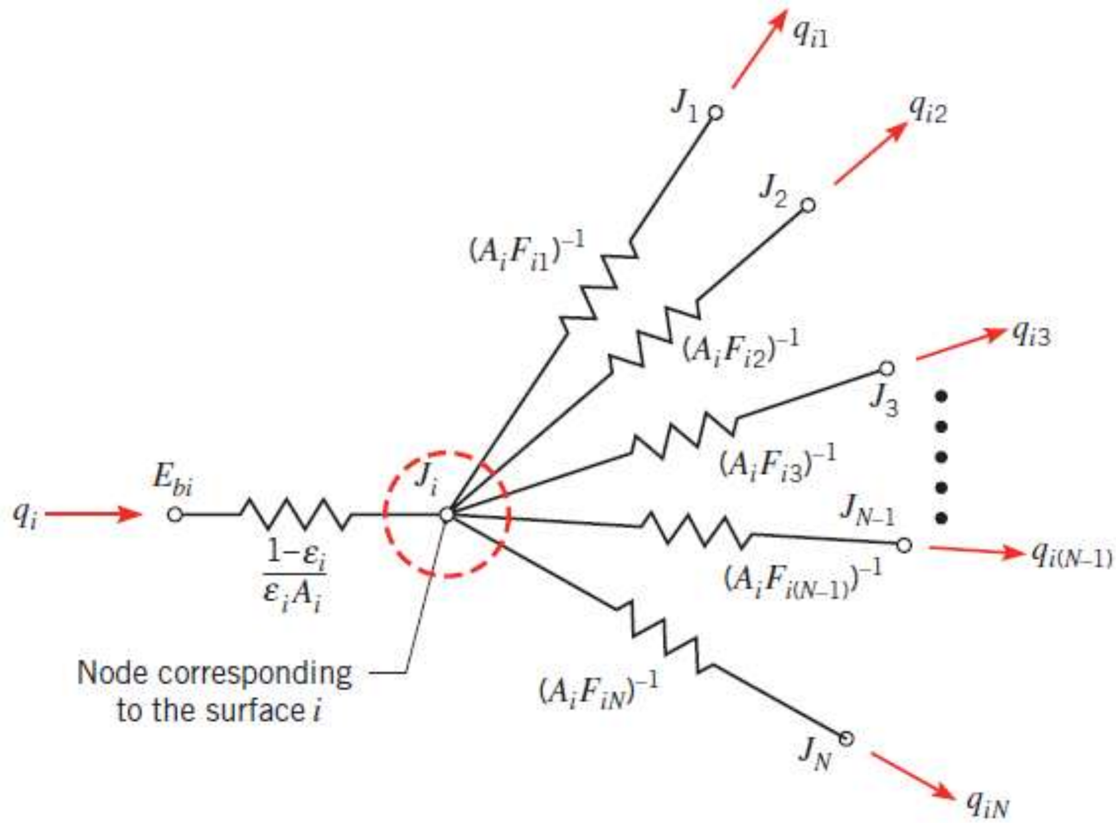
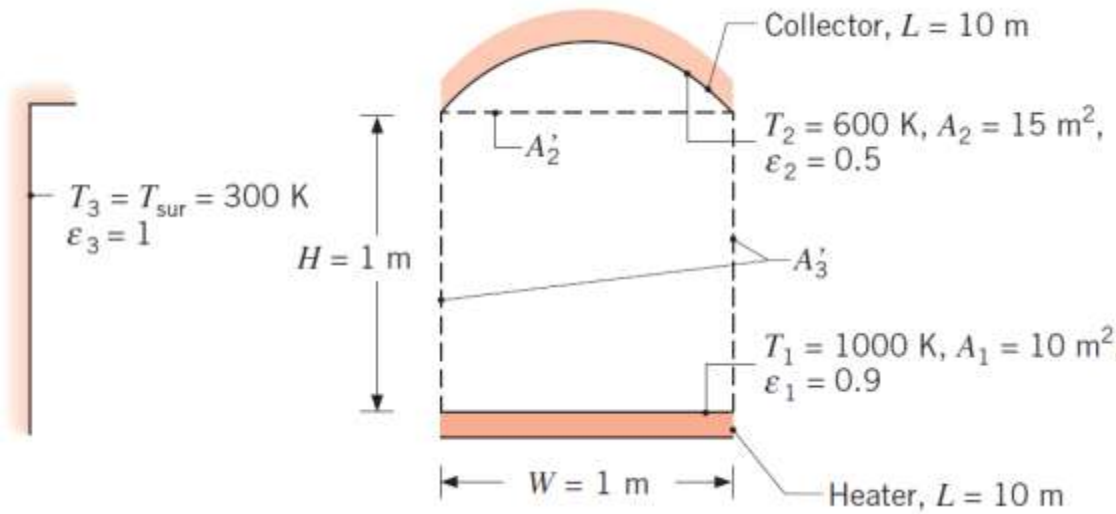
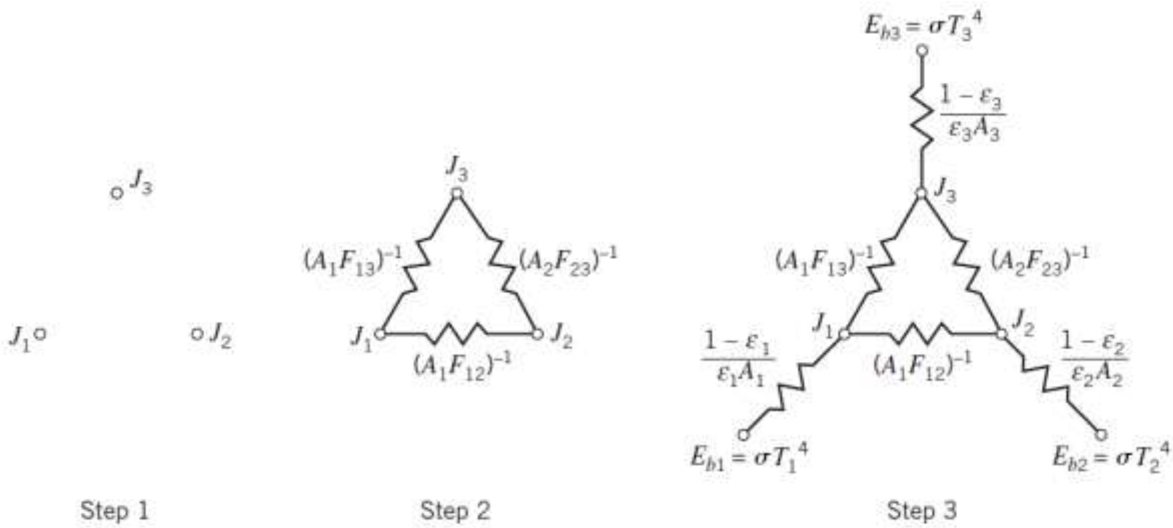


Figure 6.18: Network representation of radiative exchange between surface i and the remaining surfaces of an enclosure.

Example 4: In manufacturing, the special coating on a curved solar absorber surface of area A_2= 15 m^2 is cured by exposing it to an infrared heater of width W = 1 m. The absorber and heater are each of length L = 10 m and are separated by a distance of H = 1 m. The upper surface of the absorber and the lower surface of the heater are insulated. The heater is at T_1= 1000 K and has an emissivity of ε_1= 0.9, while the absorber is at T_2= 600 K and has an emissivity of ε_2 = 0.5. The system is in a large room whose walls are at 300 K. What is the net rate of heat transfer to the absorber surface?

Solution:





The radiation network is constructed by first identifying nodes associated with the radiosities of each surface, as shown in step 1 in the following schematic. Then each radiosity node is connected to each of the other radiosity nodes through the appropriate space resistance, as shown in step 2. We will treat the surroundings as having a large but unspecified area, which introduces difficulty in expressing the space resistances $(A_3F_{31})^{-1}$ and $(A_3F_{32})^{-1}$. Fortunately, from the reciprocity relation, we can replace A_3F_{31} with A_1F_{13} and A_3F_{32} with A_2F_{23}, which are more readily obtained. The final step is to connect the blackbody emissive powers associated with the temperature of each surface to the radiosity nodes, using the appropriate form of the surface resistance.

In this problem, the surface resistance associated with surface 3 is zero according to assumption 4; therefore, $J_3 = E_{b3} = \sigma T_3^4 = 459\ \text{W/m}^2$.

Summing currents at the J_1 node yields

$$\frac{\sigma T_1^4 - J_1}{(1-\varepsilon_1)/\varepsilon_1 A_1} = \frac{J_1 - J_2}{1/A_1 F_{12}} + \frac{J_1 - \sigma T_3^4}{1/A_1 F_{13}} \qquad (1)$$

while summing the currents at the J_2 node results in

$$\frac{\sigma T_2^4 - J_2}{(1-\varepsilon_2)/\varepsilon_2 A_2} = \frac{J_2 - J_1}{1/A_1 F_{12}} + \frac{J_2 - \sigma T_3^4}{1/A_2 F_{23}} \qquad (2)$$

The view factor F_{12} may be obtained by recognizing that $F_{12} = F_{12'}$, where A_2' is shown in the schematic as the rectangular base of the absorber surface. Then, from Figure 13.4 or Table 13.2, with $Y/L = 10/1 = 10$ and $X/L = 1/1 = 1$,

$$F_{12} = 0.39$$

From the summation rule, and recognizing that $F_{11} = 0$, it also follows that

$$F_{13} = 1 - F_{12} = 1 - 0.39 = 0.61$$

The last needed view factor is F_{23}. We recognize that, since radiation propagating from surface 2 to surface 3 must pass through the hypothetical surface A_2',

$$A_2 F_{23} = A_2' F_{2'3}$$

and from symmetry $F_{2'3} = F_{13}$. Thus

$$F_{23} = \frac{A_2'}{A_2} F_{13} = \frac{10\ \text{m}^2}{15\ \text{m}^2} \times 0.61 = 0.41$$

We may now solve Equations 1 and 2 for J_1 and J_2. Recognizing that $E_{b1} = \sigma T_1^4 = 56{,}700\ \text{W/m}^2$ and canceling the area A_1, we can express Equation 1 as

$$\frac{56{,}700 - J_1}{(1-0.9)/0.9} = \frac{J_1 - J_2}{1/0.39} + \frac{J_1 - 459}{1/0.61}$$

or

$$-10J_1 + 0.39J_2 = -510{,}582 \qquad (3)$$

Noting that $E_{b2} = \sigma T_2^4 = 7348\ \text{W/m}^2$ and dividing by the area A_2, we can express Equation 2 as

$$\frac{7348 - J_2}{(1-0.5)/0.5} = \frac{J_2 - J_1}{15\ \text{m}^2/(10\ \text{m}^2 \times 0.39)} + \frac{J_2 - 459}{1/0.41}$$

$$0.26J_1 - 1.67J_2 = -7536 \qquad (4)$$

Solving Equations 3 and 4 simultaneously yields $J_2 = 12{,}487\ \text{W/m}^2$.

An expression for the net rate of heat transfer *from* the absorber surface, q_2, may be written upon inspection of the radiation network and is

$$q_2 = \frac{\sigma T_2^4 - J_2}{(1-\varepsilon_2)/\varepsilon_2 A_2}$$

resulting in

$$q_2 = \frac{(7348 - 12{,}487)\text{W/m}^2}{(1-0.5)/(0.5 \times 15\ \text{m}^2)} = -77.1\ \text{kW}$$

Hence, the net heat transfer rate *to* the absorber is $q_{net} = -q_2 = 77.1$ kW. ◁

C. The Two-Surface Enclosure

The simplest example of an enclosure is one involving two surfaces that exchange radiation only with each other. Such a two-surface enclosure is shown schematically in Figure 6.19.*a*. Since there are only two surfaces, the net rate of radiation transfer from surface 1, q_1, must equal the net rate of radiation transfer *to* surface 2, $-q_2$, and both quantities must equal the net rate at which radiation is exchanged between 1 and 2. Accordingly,

$$q_1 = -q_2 = q_{12} \qquad (6\text{-}40)$$

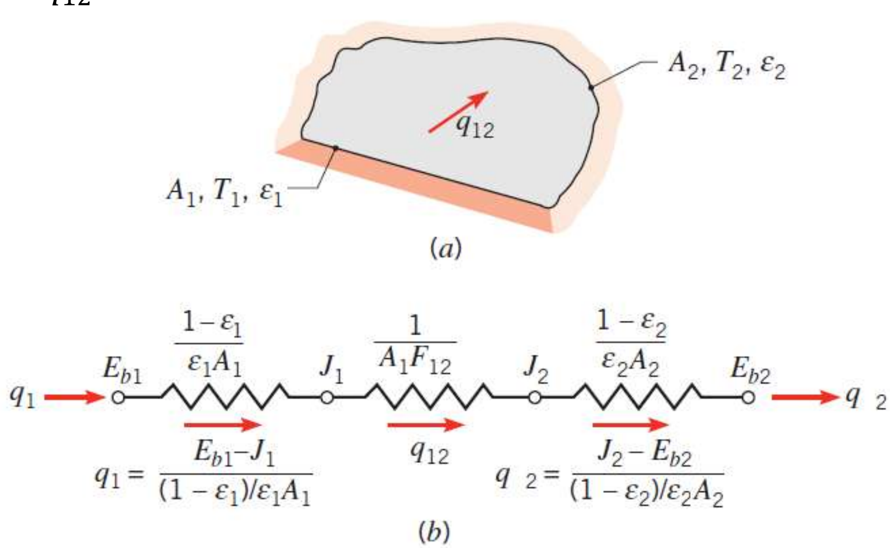


Figure 6.19: The two-surface enclosure. (a) Schematic. (b) Network representation.

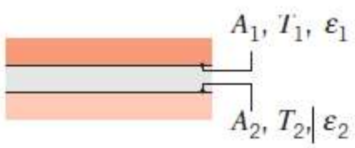
The radiation transfer rate may be determined by applying Equation 6.38 to surfaces 1 and 2 and solving the resulting two equations for J_1 and J_2. The results could then be used with Equation 6.32 to determine q_1 (or q_2). However, in this case the desired result is more readily obtained by working with the network representation of the enclosure shown in Figure 6.19.*b*. From Figure 6.19.b we see that the total resistance to radiation exchange between surfaces 1 and 2 is comprised of the two surface resistances and the geometrical resistance. Hence, substituting from Equation 12.32, the net radiation exchange between surfaces may be expressed as

$$q_{12} = q_1 = -q_2 = \frac{\sigma(T_1^4 - T_2^4)}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}} \qquad (6\text{-}41)$$

The foregoing result may be used for any two isothermal diffuse, gray surfaces that *form an enclosure* and *are each characterized by uniform radiosity and irradiation.* Important special cases are summarized in Table 6.3.

Table 6.3: Special Diffuse, Gray, Two-Surface Enclosures

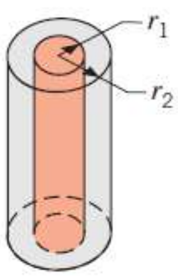
Large (Innite) Parallel Planes



$$A_1 = A_2 = A \qquad F_{12} = 1$$

$$q_{12} = \frac{A\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} - 1}$$

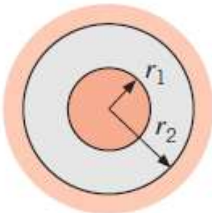
Long (Innite) Concentric Cylinders



$$\frac{A_1}{A_2} = \frac{r_1}{r_2} \qquad F_{12} = 1$$

$$q_{12} = \frac{\sigma A_1(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1-\varepsilon_2}{\varepsilon_2}\left(\dfrac{r_1}{r_2}\right)}$$

Concentric Spheres

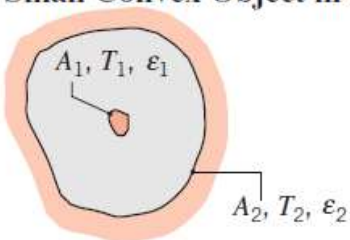


$$\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}$$

$$F_{12} = 1$$

$$q_{12} = \frac{\sigma A_1(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1-\varepsilon_2}{\varepsilon_2}\left(\dfrac{r_1}{r_2}\right)^2}$$

Small Convex Object in a Large Cavity



$$\frac{A_1}{A_2} \approx 0$$

$$F_{12} = 1$$

$$q_{12} = \sigma A_1 \varepsilon_1 (T_1^4 - T_2^4)$$

D. Radiation Shields

Radiation shields constructed from low emissivity (high reflectivity) materials can be used to reduce the net radiation transfer between two surfaces. Consider placing a radiation shield, surface 3, between the two large, parallel planes of Figure 6.20.*a*. Without the radiation shield, the net rate of radiation transfer between surfaces 1 and 2 is given by Table 6.3 for Large (Infinite) Parallel Planes. However, with the radiation shield, additional resistances are present, as shown in Figure 6.20.*b*, and the heat transfer rate is reduced. Note that the emissivity associated with one side of the shield ($\varepsilon_{3,1}$) may differ from that associated with the opposite side ($\varepsilon_{3,2}$) and the radiosities will always differ. Summing the resistances and recognizing that $F_{13} = F_{32} = 1$, it follows that

$$q_{12} = \frac{A_1\sigma(T_1^4 - T_2^4)}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} + \dfrac{1-\varepsilon_{3,1}}{\varepsilon_{3,1}} + \dfrac{1-\varepsilon_{3,2}}{\varepsilon_{3,2}}} \qquad (6\text{-}42)$$

Note that the resistances associated with the radiation shield become very large when the emissivities $\varepsilon_{3,1}$ and $\varepsilon_{3,2}$are very small.

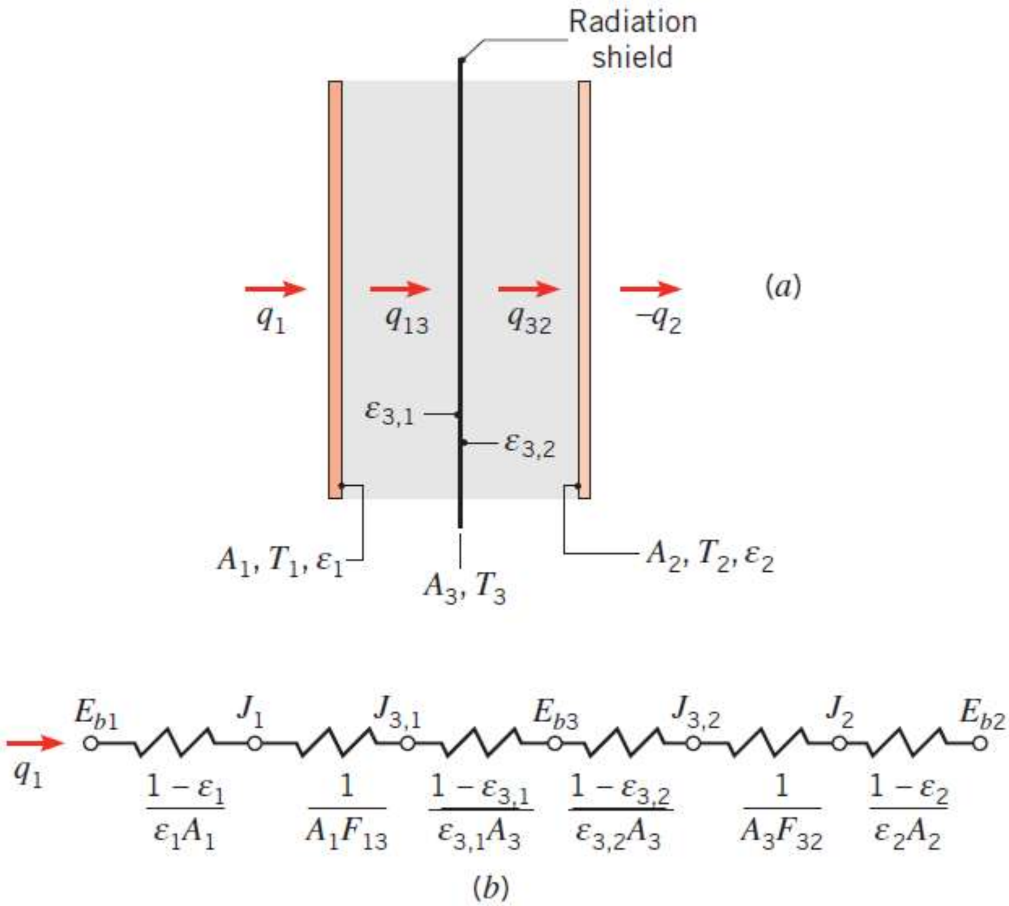


Figure 6.20: Radiation exchange between large parallel planes with a radiation shield. (a) Schematic. (b) Network representation.

Equation 6.42 may be used to determine the net heat transfer rate if T_1 and T_2 are known. From knowledge of q_{12} and the fact that $q_{12} = q_{13} = q_{32}$, the value of T_3 may then be determined by expressing Equation for Large (Infinite) Parallel Planes for q_{13} or q_{32}.

The foregoing procedure may readily be extended to problems involving multiple radiation shields. In the special case for which all the emissivities are equal, it may be shown that, with N shields,

$$(q_{12})_N = \frac{1}{N+1}(q_{12})_0 \qquad (6\text{-}43)$$

where $(q_{12})_0$ is the radiation transfer rate with no shields ($N = 0$).

Example 5:

A cryogenic fluid flows through a long tube of 20-mm diameter, the outer surface of which is diffuse and gray with ε_1 = 0.02 and T_1 = 77 K as shown in Figure 6.21. This tube is concentric with a larger tube of 50-mm diameter, the inner surface of which is diffuse and gray with ε_2 = 0.05 and T_2 = 300 K. The space between the surfaces is evacuated. Calculate the heat gain by the cryogenic fluid per unit length of tubes. If a thin radiation shield of 35-mm diameter and ε_3 = 0.02 (both sides) is inserted midway between the inner and outer surfaces, calculate the change (*percentage*) in heat gain per unit length of the tubes.

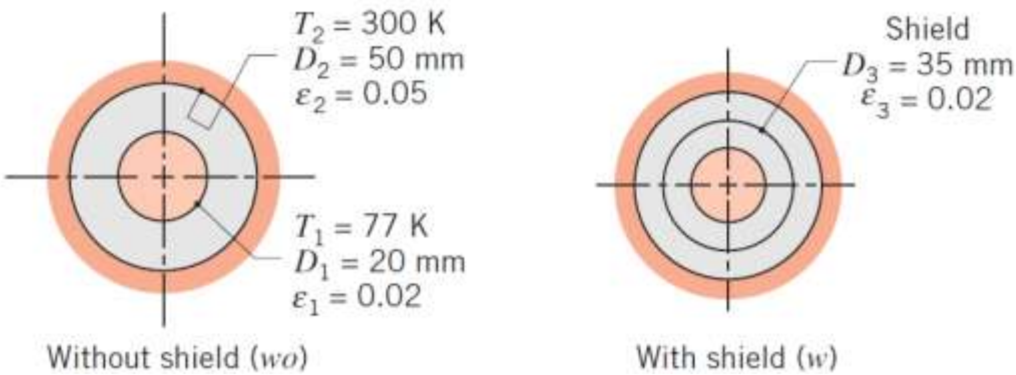


Figure 6.21: Concentric tube arrangement with diffuse, gray surfaces of different Emissivities and temperatures.

1. The network representation of the system without the shield is shown in Figure 6.19,

$$q = \frac{\sigma(\pi D_1 L)(T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1-\varepsilon_2}{\varepsilon_2}\left(\frac{D_1}{D_2}\right)}$$

Hence

$$q' = \frac{q}{L} = \frac{5.67 \times 10^{-8}\,\mathrm{W/m^2 \cdot K^4}\,(\pi \times 0.02\,\mathrm{m})[(77\,\mathrm{K})^4 - (300\,\mathrm{K})^4]}{\frac{1}{0.02} + \frac{1-0.05}{0.05}\left(\frac{0.02\,\mathrm{m}}{0.05\,\mathrm{m}}\right)}$$

$$q' = -0.50\,\mathrm{W/m}$$

2. The network representation of the system with the shield is shown in Figure 6.20, and the heat rate is now

$$q = \frac{E_{b1} - E_{b2}}{R_{\text{tot}}} = \frac{\sigma(T_1^4 - T_2^4)}{R_{\text{tot}}}$$

where

$$R_{\text{tot}} = \frac{1-\varepsilon_1}{\varepsilon_1(\pi D_1 L)} + \frac{1}{(\pi D_1 L)F_{13}} + 2\left[\frac{1-\varepsilon_3}{\varepsilon_3(\pi D_3 L)}\right] + \frac{1}{(\pi D_3 L)F_{32}} + \frac{1-\varepsilon}{\varepsilon_2(\pi D_2}$$

or

$$R_{\text{tot}} = \frac{1}{L}\left\{\frac{1-0.02}{0.02(\pi \times 0.02\text{ m})} + \frac{1}{(\pi \times 0.02\text{ m})1}\right.$$

$$+ 2\left[\frac{1-0.02}{0.02(\pi \times 0.035\text{ m})}\right] + \frac{1}{(\pi \times 0.035\text{ m})1} + \frac{1-0.05}{0.05(\pi \times 0.05\text{ n}}$$

$$R_{\text{tot}} = \frac{1}{L}(779.9 + 15.9 + 891.3 + 9.1 + 121.0) = \frac{1817}{L}\left(\frac{1}{\text{m}^2}\right)$$

Hence

$$q' = \frac{q}{L} = \frac{5.67 \times 10^{-8}\text{ W/m}^2\cdot\text{K}^4\,[(77\text{ K})^4 - (300\text{ K})^4]}{1817\text{ (1/m)}} = -0.25\text{ W/m}$$

The percentage change in the heat gain is then

$$\frac{q'_w - q'_{wo}}{q'_{wo}} \times 100 = \frac{(-0.25\text{ W/m}) - (-0.50\text{ W/m})}{-0.50\text{ W/m}} \times 100 = -50\%$$

E. The Reradiating Surface

The assumption of a *reradiating surface* is common to many industrial applications. This idealized surface is characterized by *zero* net radiation transfer ($q_i = 0$). It is closely approached by real surfaces that are well insulated on one side *and* for which convection effects may be neglected on the opposite (radiating) side. With $q_i = 0$, it follows from Equations 6.32, 6.38 and 6.27 that $G_i = J_i = E_{bi}$. Hence, if the radiosity of a reradiating surface is known, its temperature is readily determined. In an enclosure, the equilibrium temperature of a reradiating surface is determined by its interaction with the other surfaces, and it is *independent of the emissivity of the reradiating surface.*

A three-surface enclosure, for which the third surface, surface *R*, is reradiating, is shown in Figure 6.22.*a*, and the corresponding network is shown in Figure 6.22.*b*. Surface *R* is presumed to be well insulated, and convection effects are assumed to be negligible. Hence, with $q_R = 0$, the net radiation *transfer* from surface 1 must equal the net radiation

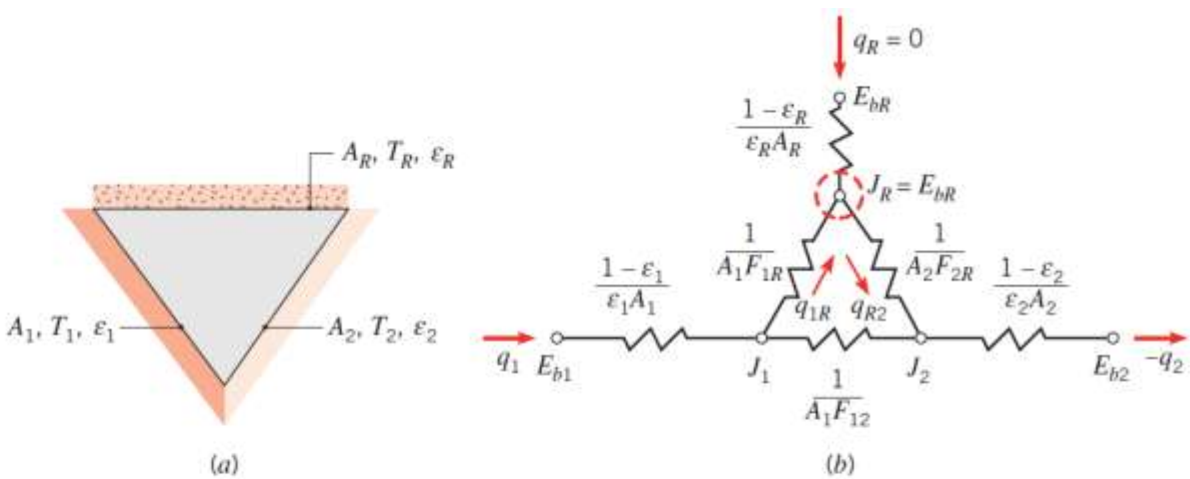


Figure 6.22: A three-surface enclosure with one surface reradiating. (*a*) Schematic. (*b*) Network representation.

transfer to surface 2. The network is a simple series–parallel arrangement, and from its analysis it is readily shown that

$$q_1 = -q_2 = \frac{E_{b1} - E_{b2}}{\frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{12} + [(1/A_1 F_{1R}) + (1/A_2 F_{2R})]^{-1}} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2}} \quad (6\text{-}44)$$

Knowing $q_1 = -q_2$, Equations 6.32 and 6.38 may be applied to surfaces 1 and 2 to determine their radiosities J_1 and J_2. Knowing J_1, J_2, and the geometrical resistances, the radiosity of the reradiating surface J_R may be determined from the radiation balance

$$\frac{J_1 - J_R}{(1/A_1 F_{1R})} - \frac{J_R - J_2}{(1/A_2 F_{2R})} = 0 \quad (6\text{-}45)$$

The temperature of the reradiating surface may then be determined from the requirement that $\sigma T_R^4 = J_R$.

Example 6:
A paint baking oven consists of a long, triangular duct in which a heated surface is maintained at 1200 K and another surface is insulated as shown in Figure 6.23. Painted panels, which are maintained at 500 K, occupy the third surface. The triangle is of width W =1 m on a side, and the heated and insulated surfaces have an emissivity of 0.8. The emissivity of the panels is 0.4. During steady-state operation, at what rate must energy be supplied to the heated side per unit length of the duct to maintain its temperature at 1200 K? What is the temperature of the insulated surface?

Solution:

1. The system may be modeled as a three-surface enclosure with one surface reradiating. The rate at which energy must be supplied to the heated surface may then be obtained from Equation 6.44:

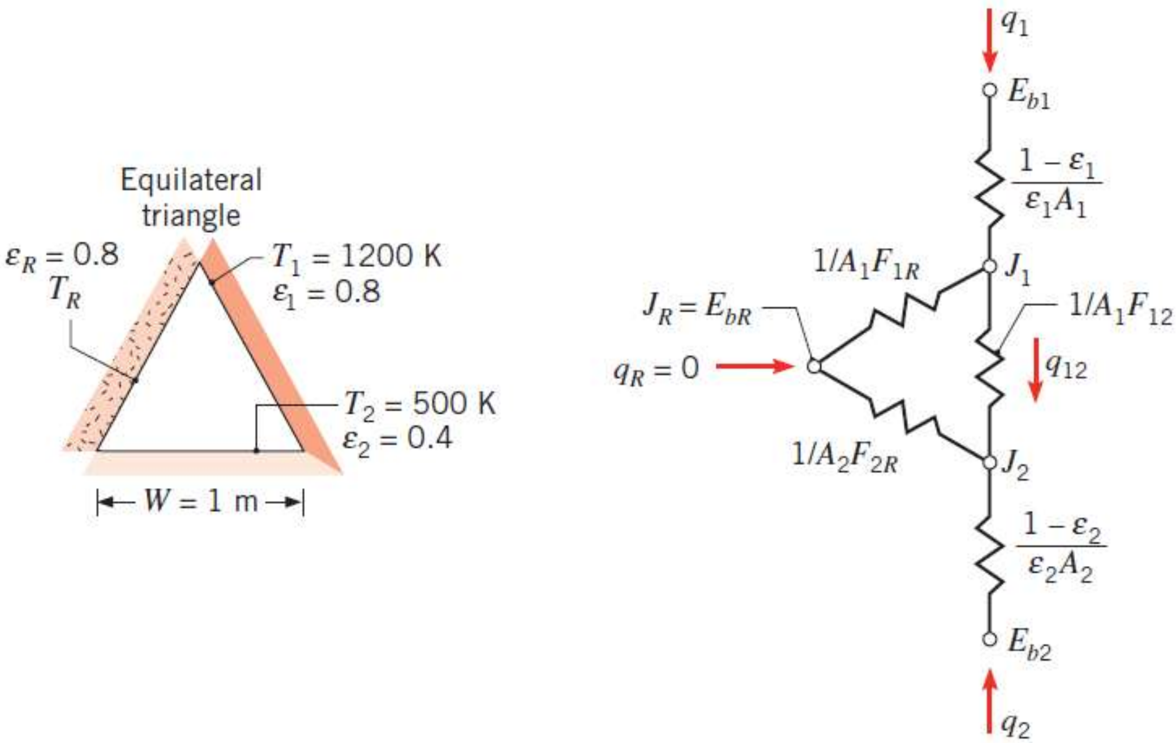


Figure 6.23: Surface properties of a long triangular duct that is insulated on one side and heated and cooled on the other sides

$$q_1 = \frac{E_{b1} - E_{b2}}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1F_{12} + [(1/A_1F_{1R}) + (1/A_2F_{2R})]^{-1}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

From symmetry, $F_{12} = F_{1R} = F_{2R} = 0.5$. Also, $A_1 = A_2 = W \cdot L$, where L is the duct length. Hence

$$q_1' = \frac{q_1}{L} = \frac{5.67 \times 10^{-8}\,\text{W/m}^2\cdot\text{K}^4\,(1200^4 - 500^4)\,\text{K}^4}{\dfrac{1-0.8}{0.8 \times 1\,\text{m}} + \dfrac{1}{1\,\text{m} \times 0.5 + (2+2)^{-1}\,\text{m}} + \dfrac{1-0.4}{0.4 \times 1\,\text{m}}}$$

or

$$q_1' = 37\,\text{kW/m} = -q_2' \qquad \triangleleft$$

2. The temperature of the insulated surface may be obtained from the requirement that $J_R = E_{bR}$, where J_R may be obtained from Equation 6.44.

However, to use this expression J_1 and J_2 must be known. Applying the surface energy balance, Equation 6.32, to surfaces 1 and 2, it follows that

$$J_1 = E_{b1} - \frac{1-\varepsilon_1}{\varepsilon_1 W} q_1' = 5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}\ (1200\ \mathrm{K})^4$$

$$- \frac{1-0.8}{0.8 \times 1\ \mathrm{m}} \times 37{,}000\ \mathrm{W/m} = 108{,}323\ \mathrm{W/m^2}$$

$$J_2 = E_{b2} - \frac{1-\varepsilon_2}{\varepsilon_2 W} q_2' = 5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}\ (500\ \mathrm{K})^4$$

$$- \frac{1-0.4}{0.4 \times 1\ \mathrm{m}} (-37{,}000\ \mathrm{W/m}) = 59{,}043\ \mathrm{W/m^2}$$

From the energy balance for the reradiating surface, Equation 13.31, it follows that

$$\frac{108{,}323 - J_R}{\dfrac{1}{W \times L \times 0.5}} - \frac{J_R - 59{,}043}{\dfrac{1}{W \times L \times 0.5}} = 0$$

Hence

$$J_R = 83{,}683\ \mathrm{W/m^2} = E_{bR} = \sigma T_R^4$$

$$T_R = \left(\frac{83{,}683\ \mathrm{W/m^2}}{5.67 \times 10^{-8}\ \mathrm{W/m^2 \cdot K^4}} \right)^{1/4} = 1102\ \mathrm{K}$$

F. Multimode Heat Transfer

Thus far, radiation exchange in an enclosure has been considered under conditions for which conduction and convection could be neglected. However, in many applications, convection and/or conduction are comparable to radiation and must be considered in the heat transfer analysis.

Consider the general surface condition of Figure 6.24.*a*. In addition to exchanging energy by radiation with other surfaces of the enclosure, there may be external heat addition to the surface, as, for example, by electric heating, and heat transfer from the surface by convection and conduction. From a surface energy balance, it follows that

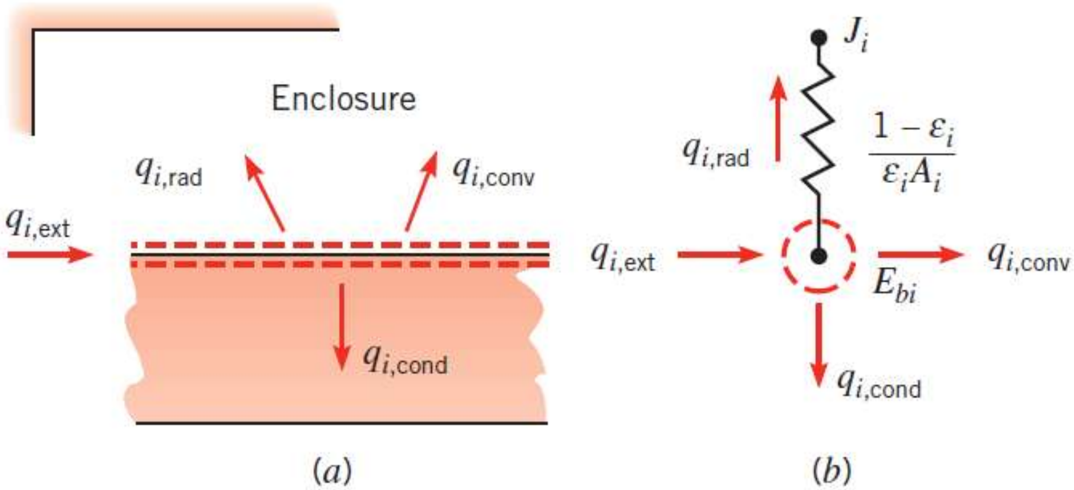


Figure 6.24: Multimode heat transfer from a surface in an enclosure. (*a*) Surface energy balance. (*b*) Circuit representation.

$$q_{i,\text{ext}} = q_{i,\text{rad}} + q_{i,\text{conv}} + q_{i,\text{cond}} \qquad (6\text{-}46)$$

where $q_{i,\text{ext}}$, $q_{i,\text{cond}}$, and $q_{i,\text{conv}}$ represent current flows to or from the surface node. Note, however, that while $q_{i,\text{cond}}$ and $q_{i,\text{conv}}$ are proportional to temperature differences, $q_{i,\text{rad}}$ is proportional to the difference between temperatures raised to the fourth power. Conditions are simplified if the back of the surface is insulated, in which case $q_{i,\text{cond}} = 0$. Moreover, if there is no external heating and convection is negligible, the surface is reradiating.